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# **A Comparative Case Study of Dynamic Models for DO-BOD- ALGAE Interaction in a Freshwater River**

**Beck, M.B.**

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**A COMPARATIVE CASE STUDY OF DYNAMIC MODELS FOR  
DO—BOD—ALGAE INTERACTION IN A FRESHWATER RIVER**

**M.B. Beck**

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## PREFACE

Over the next few years various Tasks of the Resources and Environment (REN) Area at IIASA will concentrate on selected problems of water resources management, ecology, and environmental quality control. Aspects of water resources management have been studied at IIASA since its inception, although only recently has the scope of these studies been extended to include more detailed analysis of the *quality* of water resources. One of the objectives of the current Task 2 of Resources and Environment, "Models for Environmental Quality Control and Management," is the *development* and *application* of models for analyzing the impact of waste discharges on the hydrophysical and ecological processes taking place in aquatic environments.

This paper, one of the first to report on the activities of Task 2, is concerned with the subject of *river water quality model development*. The paper summarizes and compares earlier extensive analyses of experimental time-series field data from a lowland river in England. In this latter sense the paper stands between publications originating from other past and prospective IIASA studies: the forthcoming McGraw-Hill publication *Modelling and Control of River Quality* discusses in detail some of the results presented here—the book is a product of a project jointly supported by the Centro Teoria dei Sistemi CNR, Milan, Italy, and IIASA; secondly, the summarizing nature of the paper overlaps with Task 2's objectives for the preparation and publication of a *survey of water quality modeling*.

Some of the reasons for Task 2's state-of-the-art survey include the desire to clarify the capabilities of water quality models and to accelerate the transfer of existing modeling technologies. It is not the intention of this paper to assist in the transfer of a packaged software for water quality models, even though a number of computational notes are included and, in principle, the models are ready for management applications. Rather, we hope that this paper will facilitate the transfer of field data for the evaluation of water quality models. And we hope that the field data will prove to be educational in the development of software and algorithms for identification and parameter estimation, since these are some of the basic tools of systems analysis in model building.

The subject of model applications in the context of operational river basin management is discussed in Beck (1978c).



## SUMMARY

From recent IIASA workshops on water quality modeling a need can be identified for comparative studies of different model types against the same set of field data. Similarly, some of the motivation for a state-of-the-art survey on water quality modeling to be prepared under the auspices of IIASA stems from the desire to bring order and authenticity to a fast-developing field of technology. The problem is as follows: although models can be readily applied in management and decisionmaking, they are not always so readily subject to a prior verification against field data from the river system. One reason underlying this problem is that the relevant field data, with a sufficiently high sampling frequency and collected over a sufficiently long period, either do not exist or have not been publicized.

The primary objective of this paper is the dissemination of a set of time-series field data suitable for the identification and verification of *dynamic* models for water quality. Here water quality is interpreted as the interaction between three variables: dissolved oxygen (DO) concentration—a broad measure of the healthy state, or otherwise, of a river; biochemical oxygen demand (BOD) concentration—a macro-measure of typical municipal/domestic organic waste materials; and a population of algae. A secondary objective is the comparison, by means of response error statistics, of several models that have been derived by reference to the field data. And yet a third objective is to present a summarizing and concluding statement on river water quality model development exercise which spans various publications over the past four or five years.

With respect to model comparison and model assessment the paper concludes with a cautionary message on the use of simple fitting error statistics; and, in any case, it is argued that judgments about the “best” model are dependent upon the intended application of the model. On the accuracy of the models as representations of the real system it is found that many questions remain unresolved, and particularly so for those aspects of the models related to the growth kinetics, death, and decay properties of floating algal populations. The hope is expressed that, given the data, others will be stimulated not only to answer these questions but also to reassess the assumptions that the paper makes concerning the mixing and transport characteristics of the case study reach of river.





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## 1. INTRODUCTION

Over the past ten years many models for river water quality have been developed. A substantial proportion of this modeling effort has been concerned with understanding and quantifying the relationships between stream dissolved oxygen (DO) concentration and biochemical oxygen demand (BOD) concentration. Recently these DO-BOD models have been extended to embrace more detailed relationships between various ecological constituents that characterize the quality of a water body, thus providing the potential for more sophisticated assessments of the impact of waste discharges on an aquatic environment. Not all of the models proposed so far, however, have had the benefit of being verified against field data; thus any decision maker or manager requiring the application of a water quality model might justifiably be sceptical and confused at the variety of models available to him.

The purpose of this paper is to offer a vehicle for such model verification and model comparison studies through the publication of a suitable set of field data. A secondary aim of the paper is to catalogue a number of models that have been derived by reference to this field data set and hence to conclude a project that has now been extended--quite beyond its original expectations--to a period of some five years. From the models presented it will be evident that there is a significant gap in the analysis. No form of partial differential equation, advection-diffusion model has been tested with the data, and it is hoped that others will be encouraged to complete this section of the analysis. Since this latter class of models is quite general in nature it would seem to be a straightforward matter to deduce the conditions necessary for their application to the reach of river in question.

The format of the paper is as follows. Section 2 briefly describes the characteristics of the data and river system. The data comprise a set of time series for daily sampled values of DO concentration, BOD concentration, discharge, temperature, and sunlight conditions; they refer to a short stretch (4.5 km) of the River Cam in England for the summer period of 1972. Section 3 complements section 2 by defining the nature and notation of the classes of models to be analyzed and by formalizing a simple statistical criterion for model comparison. At this point the assumptions underlying the derivation of *ordinary* differential equation forms for dynamic models of DO-BOD interaction are restated (see also Beck and Young, 1975). These assumptions are an important distinguishing feature of the modeling approach that has been adopted. Broadly speaking there are two classes of models of interest, namely internally descriptive (mechanistic) models, examples of

which are given in section 4, and black box (input/output) models, examples of which are given in section 5. A discussion of model structure identification and parameter estimation for each individual model, however, is not included; nor is there any discussion of model application (e.g., in operational control situations), and interpretations on the significance and forms of the models are kept to a minimum. Section 6 of the paper summarizes some potentially controversial issues connected with the data and modeling studies and poses certain open questions: questions on the method of model assessment and on the biochemical/ecological accuracy of the models. Although "fitting error" statistics are defined and used throughout the paper, no conclusion is made about which is the "best" model, since this kind of judgment depends strongly upon the objectives for the intended application of the model.

The field data are listed in Appendix 1. Other appendixes contain data on the geometry of the river, additional estimation results and statistics, a description of the simulation of a time-variable transportation delay function, and an abbreviated directory of previously published material supporting, interpreting, and applying the results of the main body of the text.

## 2. INTRODUCTORY DESCRIPTION OF THE RIVER SYSTEM AND FIELD DATA

The River Cam, a tributary of the Great Ouse River, flows approximately southwest to northeast across eastern England (see Figure 1). The upper reaches of its catchment area are predominantly chalky and by the time the Cam passes through Cambridge it is already a slowly moving lowland river. Upstream of Cambridge the river carries a light loading of treated industrial (pharmaceutical, fertilizer production) and municipal effluent but is still considered suitable for bathing and recreational purposes. Just downstream of Cambridge the city discharges its sewage to the river and for some distance thereafter the stream water quality is substantially degraded. The sewage receives both primary and secondary treatment prior to discharge.

Figure 2 shows the precise definition and location of the experimental reach of river with respect to the sewage outfall. Attached weed and plant growth in this section of the river is significant, although the growth is frequently cropped during the summer for reasons of the considerable use made of the river by pleasure craft. The whole of the Cam's subcatchment is an intensive agricultural area. The land adjacent to the stretch of river in Figure 2 can be classified as fenland and is drained by a system of dykes whose water is from time to time pumped up into the river. One such dyke is situated about 30 m downstream of the lower weir in Figure 2. From the physical character of the system, therefore, significant local surface runoff or seepage into the river is unlikely; in addition no major tributary joins the river between the two weirs of Figure 2.

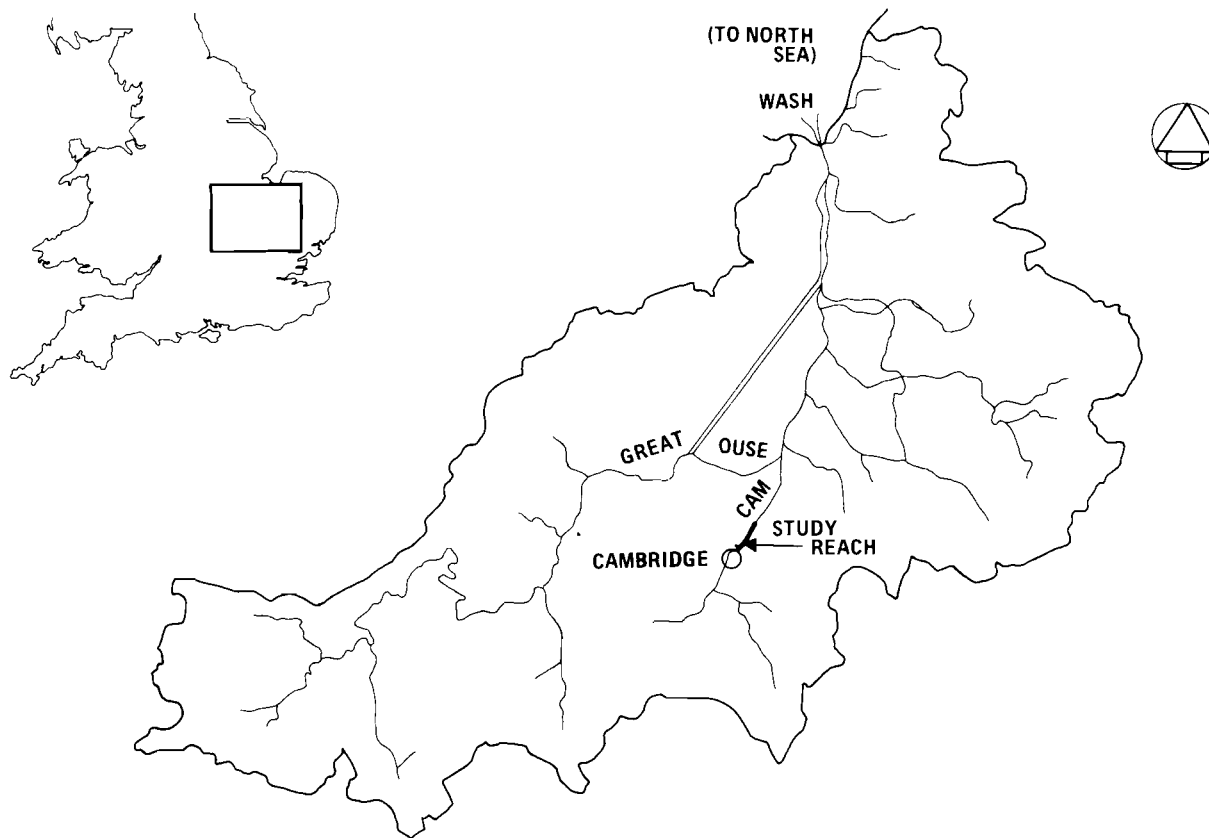


Figure 1. Location of the study reach on the River Cam. (Inset shows position of the Great Ouse Basin in relation to England and Wales.)

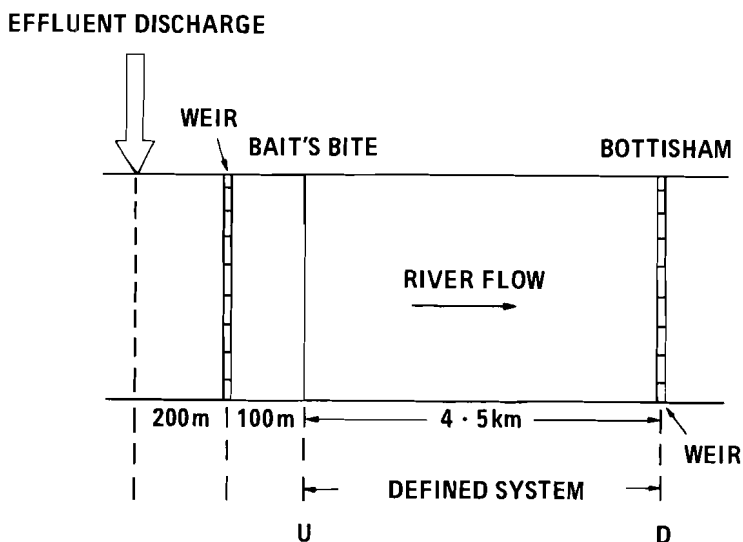


Figure 2. Schematic definition of the study reach showing the location of the effluent discharge from Cambridge Sewage Works.

With respect to obtaining measurements that give a reasonably clear picture of DO-BOD interaction dynamics, the defined system has several advantages:

- The input of sewage works effluent ensures that the system is suitably "excited" (i.e., variations in DO and BOD conditions can be observed which are not attributable to either measurement error or chance disturbance of the system).
- The critical conditions of DO-sag often occur in reaches of river immediately downstream of effluent outfalls and, in this particular river, fish kills have been reported during periods of low DO levels.
- The weir below the effluent discharge point aids the assumption of complete mixing of the effluent with the stream as it enters the defined system.
- The short reach between the upper weir and the upper system boundary is a precaution against obscuring the measurements of DO by entrained bubbles and other localized fluctuations resulting from the action of the weir.

The complete set of field data (see Appendix 1) consists of 81 daily sampled values for each variable; this covers the period from June 6th until August 25th, 1972. The measurement technique

used for each variable and data specifications are summarized in Table 1. Here it should be noted that the upstream DO and temperature measurements were obtained from a battery-operated portable/submersible monitor (loaned from the Water Research Centre, Stevenage), whereas the downstream temperature and DO recordings were recovered from a permanent monitoring station belonging to the Anglian Water Authority. For the data of Appendix 1, sampled values at 12.00 hrs each day were read from the strip-chart records from both types of monitor. The downstream monitor withdrew its sample from a median point in the river cross section; the upstream monitor was sampled at a point 4 m from one bank and at a depth of 1 m. The BOD measurements were taken at times varying between 09.00 and 15.00 hrs on any given day with the sample being drawn from the center of the respective river cross section at a depth of approximately 0.5 m.

Table 1. Summary of data specifications.

Variable	Location--With Respect to Figure 2	Sampling Rate	Measurement Technique
DO	U,D	Continuous	Monitor
BOD (five-day)	U,D	Once per day	Single grab sample
Temperature	U,D	Continuous	Monitor
Discharge	D	Once per day	Level-discharge relationship at weir
Hours of sunlight	*	Once per day	-
Rainfall	*	Once per day	-

\*Meteorological measurements were taken from a location some 8 km distant from the experimental stretch of river.

In Appendix 1 certain simplifications have therefore been made: (i) the sampling times for the BOD measurements are averaged at 12.00 hrs for each day; (ii) since no significant difference could be detected in the upstream and downstream temperature measurements only the downstream record is quoted for use in the modeling exercise. Should the reader so wish, precise sampling times for the BOD measurements and three-hourly sampled values of DO and temperature, together with daily flow-rate and (five-day) BOD measurements for the effluent discharge, are available from the author for more detailed simulation purposes. However, such information is not essential to the present discussion. In Appendix 2 a set of cross-sectional area measurements are given for regularly spaced intervals along the case study reach of river.

### 3. PRELIMINARIES: MODELS AND A METHOD OF MODEL ASSESSMENT

We shall distinguish between two classes of models. The first, denoted by the term *internally descriptive model*, is a description of the system's dynamic behavior which embodies substantial a priori knowledge of the physical, chemical, biological, and ecological phenomena governing the relationships between input, state, and output variables. The other type of model, the *black box model*, requires no such a priori information, makes no such claim to describe the internal mechanisms of the system, and is simply an empirically, or statistically, defined relationship between the observed input and output behavior.

#### 3.1 Internally Descriptive Model Definition

Figure 3a gives a schematic definition of the reach of river and some notational conventions for the measured variables. Figure 3b shows the transportation delay/continuously stirred tank

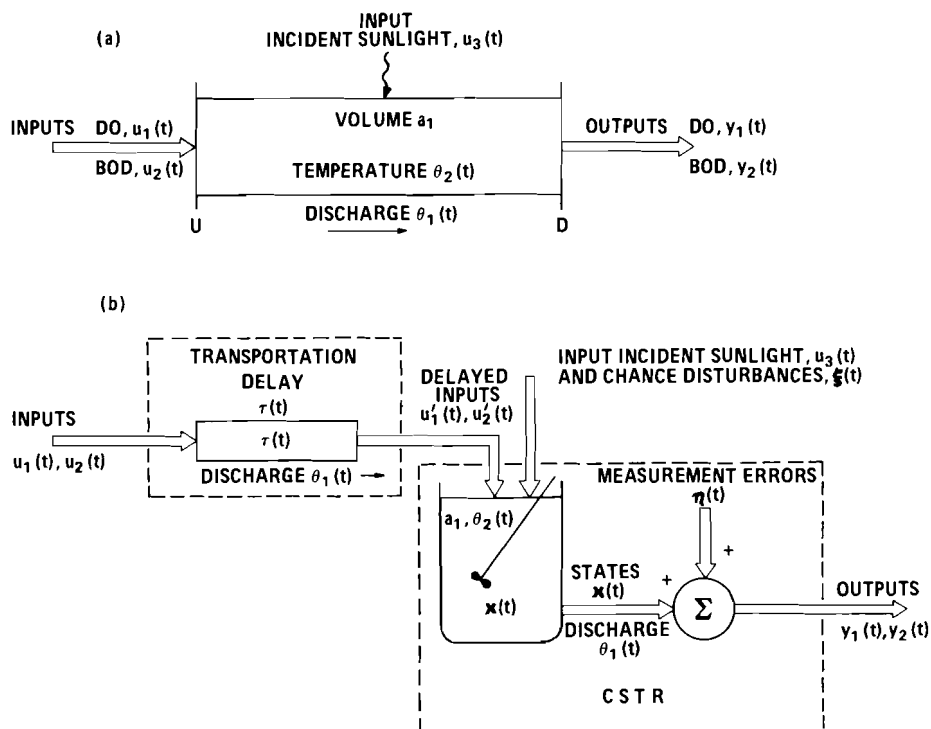


Figure 3. (a) Study reach of river with some notational conventions for the measured variables; (b) transportation delay and continuously stirred tank reactor (CSTR) idealization of the reach of river.



reactor (CSTR) *idealization* of the reach of river that permits the subsequent mathematical description of system behavior in terms of lumped-parameter, ordinary differential equation forms. Clearly this idealization draws upon standard elements of chemical engineering reactor analysis, e.g., Himmelblau and Bischoff (1968), Buffham and Gibilaro (1970); the idealization can be shown to approximate both experimentally observed transport and dispersion mechanisms (Whitehead and Young, 1975; Whitehead, 1978) and the analytical properties of distributed-parameter, partial differential equation representations of advection-diffusion mass transport (Rinaldi et al., 1978).

The reasons for the transformation of the process model from a description with time and space as the independent variables, which is intuitively more natural, to a description with just time as the single independent variable are threefold:

- The transformation simplifies subsequent computation and analysis, since, in principle, ordinary differential equations are more easily solved than partial differential equations.
- Statistical procedures for model structure identification, parameter estimation, and model verification are in practice largely restricted to lumped-parameter representations--the corresponding treatment of distributed-parameter systems is considerably less well established or understood.
- With a view to the (originally) intended application of the model for operational control purposes (Young and Beck, 1974), the vast majority of control system synthesis methods are devoted to process dynamic characterizations in terms of time as the single independent variable.

As we shall see, even with such a potentially simplifying transformation the simulation of the transportation delay element of the idealization in Figure 3b presents difficulties.

A set of component mass balances across the two elements of Figure 3b yields the following form of continuous-time, internally descriptive model.

For the CSTR

$$\dot{\underline{x}}(t) = (\theta_1(t)/a_1)\underline{u}'(t) - (\theta_1(t)/a_1)\underline{x}(t) + \underline{s}(t) + \underline{\xi}(t) \quad . \quad (1)$$

For the transportation delay

$$\underline{u}'(t) = \underline{u}(t - \tau(t)) \quad . \quad (2)$$

The general notation of Equations (1) and (2) is defined in Table 2; in specific terms:

- $\underline{u}(t)$  = vector of influent, upstream component concentrations ( $\text{gm}^{-3}$ );
- $\underline{u}'(t)$  = hypothetical time-delayed vector of influent, upstream component concentrations ( $\text{gm}^{-3}$ );
- $\underline{x}(t)$  = vector of effluent, downstream component concentrations ( $\text{gm}^{-3}$ );
- $\underline{S}(t)$  = vector of component source and sink terms ( $\text{gm}^{-3}\text{day}^{-1}$ );
- $\underline{\xi}(t)$  = vector of chance, random disturbances ( $\text{gm}^{-3}\text{day}^{-1}$ );
- $\theta_1(t)$  = stream discharge ( $\text{m}^3\text{day}^{-1}$ );
- $a_1$  = constant volume of water in defined reach of river ( $\text{m}^3$ )\*;
- $\tau(t)$  = magnitude of transportation delay element (day);
- $t$  = independent variable of time (day).

There are three basic assumptions required to combine Equations (1) and (2) in order to give the form of the internally descriptive model that is discussed subsequently:

- Assumption (1): that the volume, or depth of water in the reach of river,  $a_1$ , is constant.*
- Assumption (2): that there is no interaction between variables in the transportation delay element of the process idealization.*
- Assumption (3): that equation (2) can be approximated for this particular case study by*

$$\underline{u}'(t) = \underline{u}(t) \quad (3)$$

*which implies that  $\tau(t) = 0$  for all  $t$ .*

Assumption (1) has already been incorporated into the statement of the component mass balances of the CSTR, Equation (1); and Assumption (2) is implicit in Equation (2) in the sense that any physical, chemical, or biochemical reactions are assumed to take place only in the CSTR.

Assumption (3) is both crucial and much more difficult to justify. We may note in passing that had this assumption not been made then the description of  $\tau(t)$  merits some thought since this description needs to be time varying according to variations in the stream discharge  $\theta_1(t)$ . Methods for simulating such a

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\*In the rest of the paper, the omission of the argument  $t$  from any parameter definition indicates the assumption that the parameter is time invariant.

Table 2. Summary of general notation and variable definitions.

Variable	Definition
<b>General</b>	
$\underline{u}$	Vector of measured input variables
$\underline{x}$	Vector of state variables or hypothetical noise-free output variables
$\underline{y}$	Vector of measured output variables
<b>Internally Descriptive Model</b>	
$\underline{a}$	Vector of parameters (coefficients)
$\underline{\theta}$	Vector of variables "internal" to the model but not defined as state variables
$\underline{S}$	Vector of source and sink terms related to each state variable
$\underline{\xi}$	Vector of chance, random disturbances of the system
$\underline{\eta}$	Vector of output measurement errors
<b>Black Box Model</b>	
$\underline{\alpha}$	Vector of autoregressive polynomial parameters
$\underline{\beta}$	Vector of input polynomial parameters
$\underline{v}$	Lumped noise process accounting for both random disturbances and measurement errors
<b>Model Assessment</b>	
$\hat{\underline{x}}$	Vector of (deterministic) model output predictions
$\underline{\varepsilon}$	Vector of errors between observed output and deterministic model output predictions

time-variable transportation delay are available (see e.g. Coggan and Noton, 1970 and Appendix 4) and would almost certainly be required for longer reaches of river and for time-series data in which the sampling interval is much shorter than the average detention time of the reach. On inspection Appendix 4 suggests that to include this kind of simulation for  $\tau(t)$  is merely to exchange the complexity/computational effort of a distributed-parameter model for the complexity and effort of solving an increased number of ordinary differential equations. (In fact, partly for this reason transportation delays--or "dead time"--are extremely awkward to handle in continuous-time control system design procedures; they are much more easily accommodated in the framework of discrete

time, or digital, control system synthesis techniques.) It should now be evident why Assumption (3) is important in that it permits a considerable simplification. Yet at the same time some assessment should be made of the degree of inaccuracy introduced by the assumption.

Firstly, for the short study reach of the Cam with an average detention time during the experimental period of just over one day, and given the relatively slow sampling frequency (once per day), it is not possible to observe, and hence to identify or model, the response of DO-BOD interaction to *higher* frequency, input, upstream disturbances. Moreover, as Rinaldi et al. (1978) point out, an idealization of the river reach as a CSTR *without* any transportation delay element provides in theory an approximation to the advection-diffusion representation which is relatively better in the regime of *low* frequency disturbances than it is in the spectrum of high frequency disturbances. We would thus expect the models employed here to provide very poor approximations to the downstream DO and BOD concentrations as responses to impulsive (high frequency) changes in the upstream DO and BOD concentrations. On the other hand, with Assumption (3) the models should simulate quite well the advective transport of material downstream when conditions at the upstream boundary are changing in the manner of longer-term trends and slow periodic fluctuations, i.e., low frequency input disturbances.

Secondly, the following qualifications apply to the above kinds of argument:

- That for the integration of Equation (1) over the time interval of one day  $u(t)$  is substituted by the values measured at the *beginning* of that period (see section 4.1)--hence, the predicted downstream concentrations at 12.00 hrs on the current day are a function of the measured upstream concentrations at 12.00 hrs on the previous day (compare with the average detention time properties of the study reach); and
- That some, if not a large proportion, of the high frequency disturbances and variations in the observed process dynamics are due to stochastic effects which thus represent a kind of irreducible minimum error that can be obtained in the following modeling exercise.

Thirdly, in order to avoid confusion, let us mention that the term "transportation delay" as defined and used here is *not* equivalent to the term "time of travel". For instance, whereas the time of travel might represent the time taken to reach the peak (or center of gravity) of the downstream response to an upstream impulse tracer disturbance, the transportation delay more closely resembles the time elapsed before any significant positive response to the impulse input is detected downstream. If an average value for the time of travel can be approximated by the

ratio  $(a_1/\theta_1(t))$ , then in general

$$\tau(t) < (a_1/\theta_1(t)) \quad .$$

Rinaldi et al. (1978) suggest one such choice for  $\tau(t)$ , which is based on an analysis of how the analytical properties of the transportation delay/CSTR model compare with the properties of another lumped-parameter approximation of the advection-diffusion, partial differential equation.

Bearing in mind these preceding considerations, and having noted that inclusion of a representation for  $\tau(t)$  according to Appendix 4 produced apparently negligible differences, Assumption (3) was made at an early stage in the analysis and has since been preserved in all the models to be presented in section 4. Thus, by Equation (3), Equations (1) and (2) can be combined to give

$$\dot{\underline{x}}(t) = (\theta_1(t)/a_1)\underline{u}(t) - (\theta_1(t)/a_1)\underline{x}(t) + \underline{S}(t) + \underline{\xi}(t) \quad ; \quad (4)$$

this equation, together with the output observations  $y_1(t_k)$ ,  $y_2(t_k)$  of *downstream* DO and BOD concentrations, respectively,

$$\begin{bmatrix} y_1(t_k) \\ y_2(t_k) \end{bmatrix} = \begin{bmatrix} x_1(t_k) \\ x_2(t_k) \end{bmatrix} + \begin{bmatrix} \eta_1(t_k) \\ \eta_2(t_k) \end{bmatrix} \quad (5)$$

is the *basic description of the internally descriptive model*. In Equation (5),

$x_1(t_k)$ ,  $x_2(t_k)$  are respectively the downstream concentration of DO and BOD at time  $t_k$  ( $\text{gm}^{-3}$ );

$\eta_1(t_k)$ ,  $\eta_2(t_k)$  are respectively chance measurement errors associated with the output observations of DO and BOD ( $\text{gm}^{-3}$ );

$t_k$  is the  $k$ th sampling instant of time where the sampling interval  $\Delta t = (t_k - t_{k-1}) = 1$  (day).

### 3.2 The Black Box Model

The black box model can briefly be formally stated as

$$\underline{x}(t_k) = A(q^{-1})\underline{x}(t_k) + \sum_{i=1}^m B_i(q^{-1})u_i(t_k) \quad , \quad (6)$$

where the scalar  $x(t_k)$ , either the downstream DO concentration ( $x_1(t_k)$ ) or the downstream BOD concentration ( $x_2(t_k)$ ), is observed only in the presence of noise,

$$y(t_k) = x(t_k) + v(t_k) \quad . \quad (7)$$

Equations (6) and (7) are the *basic description of the black box model*. In Equation (6)  $q^{-1}$  is defined as the backward shift operator,

$$q^{-1}\{x(t_k)\} = x(t_{k-1}) \text{ etc. } , \quad (8)$$

and  $A(q^{-1})$  and  $B_i(q^{-1})$  are  $n$ th order polynomials in  $q^{-1}$  defined as

$$\begin{aligned} A(q^{-1}) &= \alpha_1 q^{-1} + \alpha_2 q^{-2} + \dots + \alpha_n q^{-n} , \\ B_i(q^{-1}) &= \beta_{i0} + \beta_{i1} q^{-1} + \dots + \beta_{in} q^{-n} ; i=1,2,\dots,m . \end{aligned} \quad (9)$$

The parameters  $\alpha_i$  and  $\beta_{ij}$  are respectively elements of the vectors  $\underline{\alpha}$  and  $\underline{\beta}$  referred to in Table 2.  $v(t_k)$  denotes that the random noise component of Equation (7) is a lumped term which really covers the combined effects previously accounted for (conceptually) by  $\underline{\xi}(t)$  and  $\underline{\eta}(t_k)$  in the internally descriptive model.

Since the form of the black box model is restricted to the case of *single output (state)\* systems*, its application requires

*Assumption (4): that (for black box representations) the dynamic behavior of the downstream DO concentration can be considered independent of the dynamic behavior of the downstream BOD concentration, although not necessarily independent of the input, upstream BOD concentration variations.*

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\*There is a slight problem of terminology here; however, to all intents and purposes, "outputs" are equivalent to noise corrupted observations of the "state" variables.

### 3.3 A Simple Method of Model Assessment

The method of model assessment is indeed simple. We must first, however, specify the exact nature of a *deterministic model prediction*.

For the *internally descriptive model* such a prediction is defined as the solution at time  $t_k$  of

$$\begin{aligned}\dot{\underline{x}}(t) &= (\theta_1(t)/a_1)\underline{u}(t) - (\theta_1(t)/a_1)\underline{x}(t) + \underline{s}(t) ; \\ \underline{x}(t_0) &= \hat{\underline{x}}(t_0) ,\end{aligned}\quad (10)$$

given a set of (estimated) values for the initial conditions  $\hat{\underline{x}}(t_0)$ , the *measured* data for variables  $\underline{u}(t_k)$  and  $\underline{\theta}(t_k)$ , and estimated values for all parameters  $\underline{a}$  implicit in the form of  $\underline{s}(t)$ . Precisely how the substitutions for  $\underline{u}(t_k)$  and  $\underline{\theta}(t_k)$  are made will be defined in section 4.

For the *black box model* we have

$$\underline{x}(t_k) = A(q^{-1})\underline{x}(t_k) + \sum_{i=1}^m B_i(q^{-1})\underline{u}_i(t_k) ; \underline{x}(t_0) = \hat{\underline{x}}(t_0) ,\quad (11)$$

where  $\hat{\underline{x}}(t_0)$ ,  $\underline{u}_i(t_k)$ ,  $i=1,2,\dots,m$ , and values for the parameters  $\underline{\alpha}$  and  $\underline{\beta}$  are available.

From Equations (10) and (11) the following vector (or scalar) error quantities can be determined for the internally descriptive (or black box) models:

$$\underline{\varepsilon}(t_k) = \underline{y}(t_k) - \hat{\underline{x}}(t_k) ,\quad (12)$$

and for each such *deterministic response error* sequence,  $\varepsilon(t_k)$ , we may compute corresponding sample mean,  $\mu$ , and standard deviation,  $\sigma$ , statistics,

$$\mu = [1/(N - \delta)] \sum_{j=\delta}^{80} \varepsilon(t_j) ; \sigma = [1/(N - \delta - 1)] \sum_{j=\delta}^{80} (\varepsilon(t_j) - \mu)^2 .\quad (13)$$

The notation of Equation (13) indicates that the sampled measurements for the first day of the experiment are considered to have been taken at time  $t_0$ . Thus for all internally descriptive models  $\delta = 1$ , i.e. an error can be computed for time  $t_1$ , but for the black box models  $\delta$  is dependent upon  $n$ , the chosen order for the  $B_i(q^{-1})$  and  $A(q^{-1})$  polynomials. The reasons for this will become more evident in section 5.

We may note now that in sections 3.1 and 3.2 the stochastic aspects of the models,  $\underline{\xi}(t)$ ,  $\underline{\eta}(t_k)$ , are included simply for the purpose of completeness and for emphasizing the probabilistic framework of the modeling exercise. Further consideration of these terms is incidental to the main themes of the paper and only passing reference will be made to certain estimated forms of  $v(t_k)$  in association with the black box modeling results (see Appendix 5).

#### 4. INTERNALLY DESCRIPTIVE MODELING RESULTS

In this and the following section supporting remarks on model development and interpretation are restricted to a minimum. A sufficient body of literature already exists on the Cam-1972 modeling exercise, abstracts of which are given in Appendix 3.\*

##### 4.1 Model I (Beck and Young, 1975)

This is essentially a model based on the proposals of Dobbins (1964) and his assumptions are therefore reflected in the explicit form of  $\underline{S}(t)$ :

$$\begin{aligned}
 \text{(a) DO: } \dot{x}_1(t) &= (\theta_1(t)/a_1)u_1(t) - (\theta_1(t)/a_1)x_1(t) + a_2(\theta_3(t) \\
 &\quad - x_1(t)) - a_3x_2(t) + a_4(t) \quad ; \\
 &\hspace{25em} (14) \\
 \text{(b) BOD: } \dot{x}_2(t) &= (\theta_1(t)/a_1)u_2(t) - (\theta_1(t)/a_1)x_2(t) \\
 &\quad - a_3x_2(t) + a_5(t) \quad .
 \end{aligned}$$

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\*Conversely, if there appears to be too much computational detail, this has been included to ensure that the objective of reproducibility of results can be satisfied if necessary.



The additional variables are defined as

$$\begin{aligned} u_1(t), u_2(t) &= \text{respectively, the upstream (input) DO and} \\ &\quad \text{BOD concentrations (gm}^{-3}\text{);} \\ \theta_3(t) &= \text{saturation concentration of DO (gm}^{-3}\text{)} \end{aligned}$$

where  $\theta_3(t)$  is computed from the following relationship with the stream water temperature  $\theta_2(t)$ ,

$$\begin{aligned} \theta_3(t) &= 14.541233 - 0.3928026\theta_2(t) + 0.00732326[\theta_2(t)]^2 \\ &\quad - 0.00006629[\theta_2(t)]^3 . \end{aligned} \quad (15)$$

The initial conditions, parameter values and definitions, and error statistics for this model are given in Table 3; a comparison

Table 3. Initial conditions, parameter values, and error statistics for Model I.

Variable (Parameter)	Definition	Value
$\hat{x}_1(t_0)$	Initial conditions for downstream DO concentration	$8.0 \text{ gm}^{-3}$
$\hat{x}_2(t_0)$	Initial conditions for downstream BOD concentration	$1.4 \text{ gm}^{-3}$
$a_1$	Volumetric hold-up in the reach	$1.51 \times 10^5 \text{ m}^3$
$a_2$	Reaeration rate constant	$0.17 \text{ day}^{-1}$
$a_3$	BOD decay rate constant	$0.32 \text{ day}^{-1}$
$a_4(t)$	Net rate of addition of DO to reach by combined effects of photosynthetic/respiratory activity of plants and algae and the decomposition of mud deposits	$\left. \begin{aligned} &-2.7 \text{ for } 0 \leq t \leq t_{19} \\ &-0.4 \text{ for } t > t_{19} \end{aligned} \right\}$ $(\text{in gm}^{-3} \text{ day}^{-1})$
$a_5(t)$	Rate of addition of BOD to reach by local surface runoff	$0 \text{ for all } t$ $(\text{in gm}^{-3} \text{ day}^{-1})$
$\mu_1$	Mean of errors in DO predictions	$0.465 \text{ gm}^{-3}$
$\sigma_1$	Standard deviation of errors in DO predictions	$0.855 \text{ gm}^{-3}$
$\mu_2$	Mean of errors in BOD predictions	$0.817 \text{ gm}^{-3}$
$\sigma_2$	Standard deviation of errors in BOD predictions	$1.271 \text{ gm}^{-3}$

of the deterministic model responses  $\hat{x}(t_k)$  and observations  $y(t_k)$  is given in Figure 4.\* In Figure 4 the reader's attention is drawn to the performance of the model over the periods  $t_{36} \rightarrow t_{48}$  (both the DO and the BOD responses) and from  $t_{60}$  onwards (for the BOD response). Any significant improvement afforded by the later models will be most evident at these points in the experiment. The predicted downstream BOD concentration on day  $t_{58}$  should also be noted: it results from the effects of a thunderstorm on day  $t_{56}$ , giving rise to a peak upstream BOD concentration on day  $t_{57}$  which probably led in turn to an actual peak downstream BOD some time between the samples of  $t_{57}$  and  $t_{58}$ . This then is precisely the kind of high frequency response characteristic that we should not expect the model to be able to reproduce accurately (see section 3.1). However, it is difficult to confirm that this is so since during high flow conditions the detention time in the reach approaches a minimum value and the daily sampling frequency of the data is consequently too *slow* to pick up the fast transient responses to the impulsive disturbance of the thunderstorm.

#### Computational Note

Solutions to Equation (14) are obtained iteratively by numerical integration (Runge-Kutta) over the interval  $t_k \rightarrow t_{k+1}$ . For this interval, therefore, the values

$$\left. \begin{aligned} \underline{u}(t) &= \underline{u}(t_k) \\ \underline{0}(t) &= \underline{0}(t_k) \end{aligned} \right\} \text{ for } t_k \leq t \leq t_{k+1} \quad (16)$$

are substituted. Thus note that the alternative of linear interpolation may in fact yield more accurate results and especially so for the case of the storm conditions discussed above.

#### 4.2 Model II (Beck and Young, 1975, 1976)

Whereas Model I does not account for the interaction of an algal population with the DO and BOD dynamics, this is incorporated into Model II by means of a new pseudoempirical relationship for "sustained sunlight effects", that is

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\*See also Appendix 1 for comments on the salient features of the experimental data.

$$\begin{aligned} \text{DO: } \dot{x}_1(t) = & (\theta_1(t)/a_1)u_1(t) - (\theta_1(t)/a_1)x_1(t) + a_2(\theta_3(t) - x_1(t)) \\ & - a_3x_2(t) + a_4'(t) + a_6(\theta_4(t) - a_8) \quad ; \end{aligned} \quad (17a)$$

$$\begin{aligned} \text{BOD: } \dot{x}_2(t) = & (\theta_1(t)/a_1)u_2(t) - (\theta_1(t)/a_1)x_2(t) - a_3x_2(t) + a_5(t) \\ & + a_7(\theta_4(t) - a_8) \quad ; \end{aligned} \quad (17b)$$

where

$$\theta_4(t_k) = \theta_4(t_{k-1}) + a_9[u_3(t_k)\{(\theta_2(t_k) - a_{10})/a_{10}\} - \theta_4(t_{k-1})] \quad ; \quad (17c)$$

with

$$(\theta_4(t_k) - a_8) = 0 \text{ for } \theta_4(t_k) < a_8 \quad . \quad (17d)$$

The variables  $u_3(t_k)$  and  $\theta_4(t_k)$  are defined as

$$\begin{aligned} u_3(t_k) &= \text{hours of sunlight incident on the system at day } t_k; \\ \theta_4(t_k) &= \text{"sustained sunlight effect" at day } t_k \text{ (hours of} \\ &\quad \text{sunlight per day).} \end{aligned}$$

Figure 5 shows a significant improvement in the model responses, particularly over the period  $t_{36} \rightarrow t_{48}$ , given the additional initial conditions and parameter values listed in Table 4; the improved model performance is reflected in the error statistics also shown in Table 4. Model II requires

*Assumption (5): that the higher observed DO and BOD conditions for  $t_{36} \rightarrow t_{48}$  are due to the growth of an algal population, which in turn is some function of the cumulative influence of warm, sunny periods of weather.*

In other words the major additional hypotheses included in the model of Equation (17) are as follows:

- The low-pass filter formulation of Equation (17c) produces a time-series  $\theta_4(t_k)$  in which longer periods of

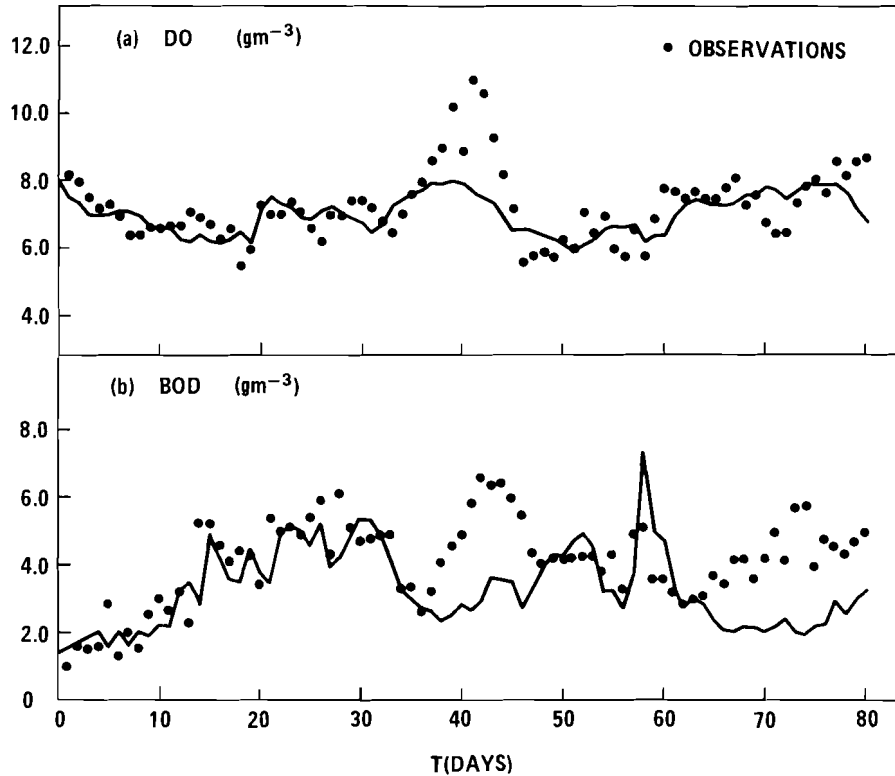


Figure 4. Deterministic model responses  $\hat{x}(t_k)$ , and observations,  $y(t_k)$ , for Model I: (a) downstream DO concentration, (b) downstream BOD concentration.

sunny weather are emphasized at the expense of the higher frequency variations such as alternate bright and dull days (hence the term, low-pass filter).

- The inequality constraint of Equation (17c) embodies the notion that a certain minimum level of sustained sunny weather is required before the algal population propagates itself to significant proportions.
- Sunlight and temperature are regarded as the controlling factors of algal growth since other nutrients, principally deriving from the sewage discharge, are assumed to be available in nonlimiting quantities.

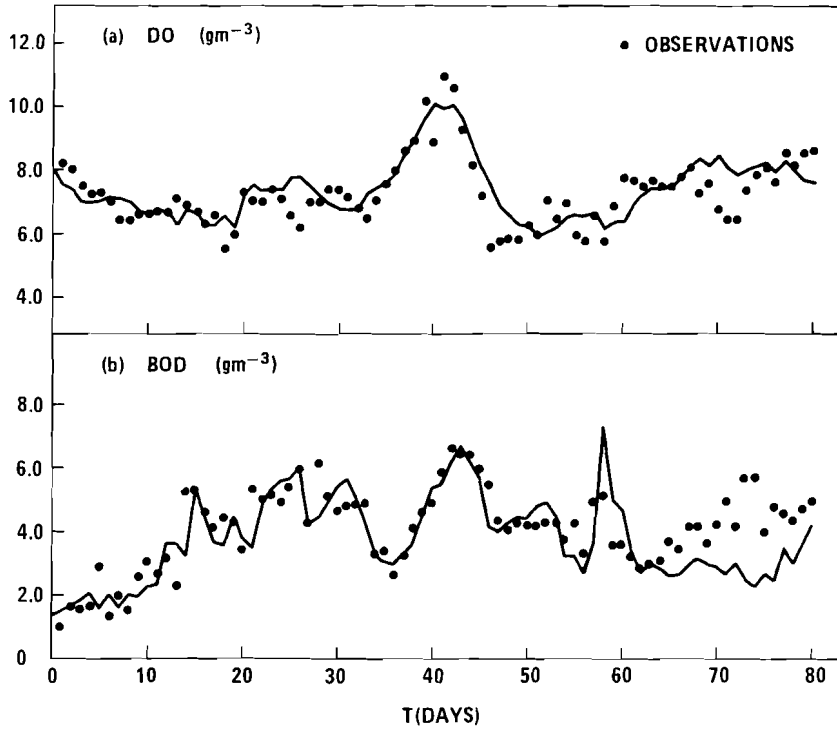


Figure 5. Deterministic model responses,  $\hat{x}(t_k)$ , and observations,  $y(t_k)$ , for Model II: (a) downstream DO concentration, (b) downstream BOD concentration.

#### Computational Note

Conditions similar to those of Equation (14) hold for the solution of Equations (17). A further condition is, in Equations (17a) and (17b),

$$\Theta_4(t) = \Theta_4(t_k) \text{ for } t_k \leq t \leq t_{k+1} \quad . \quad (18)$$

Table 4. Error statistics and additional initial conditions and parameter values for Model II.

Variable (Parameter)	Definition	Value
$\theta_4(t_0)$	Initial conditions for sustained sunlight effect	0.0 hr*
$a_4(t)$	Rate of addition of DO to reach by decomposition of bottom mud deposits	(as for $a_4(t)$ )
$a_6$	Coefficient for sustained sunlight effect in DO equation	$0.31 \text{ gm}^{-3} \text{ DO} \cdot \text{day}^{-1} \text{ hr}^{-1*}$
$a_7$	Coefficient for sustained sunlight effect in BOD equation	$0.32 \text{ gm}^{-3} \text{ BOD} \cdot \text{day}^{-1} \text{ hr}^{-1*}$
$a_8$	Threshold level for sustained sunlight effect	6.0 hr*
$a_9$	Reciprocal time constant for discrete-time low-pass filter for the sustained sunlight effect, equation (17c)	0.25 (dimensionless)
$a_{10}$	Arbitrary mean river water temperature	8.0 °C
$\mu_1$	Mean of errors in DO predictions	$0.088 \text{ gm}^{-3}$
$\sigma_1$	Standard deviation of errors in DO predictions	$0.657 \text{ gm}^{-3}$
$\mu_2$	Mean of errors in BOD predictions	$0.328 \text{ gm}^{-3}$
$\sigma_2$	Standard deviation of errors in BOD predictions	$0.968 \text{ gm}^{-3}$

\*The abbreviation hr is used to denote hours of sunlight per day.

#### 4.3 Model III (Beck 1974, 1975)

The discrete-time low-pass filter mechanism for the sustained sunlight effect in Model II, Equation (17c), has an analog continuous-time form. On the basis of certain observations (Beck, 1975) it is found to be more appropriate, however, to simulate the growth and interaction effects of an algal population by *two* low-pass filters in series:

$$\begin{aligned} \text{DO: } \dot{x}_1(t) = & (\theta_1(t)/a_1)u_1(t) - (\theta_1(t)/a_1)x_1(t) + a_2(\theta_3(t) - x_1(t)) \\ & - a_3x_2(t) + a_4'(t) + a_{11}(x_3(t) - a_{12}) \quad ; \quad (19a) \end{aligned}$$

$$\begin{aligned} \text{BOD: } \dot{x}_2(t) = & (\theta_1(t)/a_1)u_2(t) - (\theta_2(t)/a_1)x_2(t) - a_3x_2(t) + a_5(t) \\ & + a_{13}(x_4(t) - a_{14}) \quad ; \quad (19b) \end{aligned}$$

$$\dot{x}_3(t) = - (1/a_{15})x_3(t) + (a_{16}/a_{15})u_3(t) \quad ; \quad (19c)$$

$$\dot{x}_4(t) = - (1/a_{17})x_4(t) + (1/a_{17})x_3(t) \quad , \quad (19d)$$

in which  $x_3(t)$  = output of first low-pass filter (hours of sunlight per day and  $x_4(t)$  = output of second low-pass filter (hours of sunlight per day).

Notice that  $x_3(t)$  interacts only with the downstream DO concentration, while  $x_4(t)$  interacts only with the downstream BOD concentration;  $x_3(t)$  and  $x_4(t)$  therefore fulfil in Equations (19a) and (19b) the equivalent roles of  $\theta_4(t)$  in Equations (17a) and (17b).

Table 5 summarizes the parameter values, initial conditions, and deterministic response error statistics for Model III and a comparison of the model performance with the observed behavior is given in Figure 6. Model III can be seen to be only marginally "better" at fitting the data than Model II; however, Equation (19) is useful primarily as a conceptual link between Models II and IV since it indicates that the sustained sunlight effect has a more immediate influence on the DO and a slower interaction with the BOD.

#### Computational Note

The conditions of Equation (14), together with the substitution

$$u_3(t) = u_3(t_k) \text{ for } t_k \leq t \leq t_{k+1} \quad (20)$$

in Equation (19c), hold for solutions of Equations (19). The inequality constraint of Equation (17d) is not transferred in any equivalent form to Equations (19).

Table 5. Error statistics and additional initial conditions and parameter values for Model III.

Variable (Parameter)	Definition	Value
$\hat{x}_3(t_0)$	Initial conditions for output of first low-pass filter	1.0 hr*
$\hat{x}_4(t_0)$	Initial conditions for output of second low-pass filter	1.0 hr*
$a_{11}$	Coefficient for equivalent sustained sunlight effect in DO equation	$0.115 \text{ gm}^{-3} \text{ DO} \cdot \text{day}^{-1} \text{ hr}^{-1} *$
$a_{12}$	"Threshold" level for equivalent sustained sunlight effect in DO equation	6.0 hr*
$a_{13}$	Coefficient for equivalent sustained effect in BOD equation	$0.146 \text{ gm}^{-3} \text{ BOD} \cdot \text{day}^{-1} \text{ hr}^{-1} *$
$a_{14}$	"Threshold" level for equivalent sustained sunlight effect in BOD equation	6.0 hr*
$a_{15}$	Time-constant for first low-pass filter	1.95 day
$a_{16}$	Gain coefficient between $u_3(t)$ and $x_3(t)$	2.33 (dimensionless)
$a_{17}$	Time constant for second low-pass filter	1.42 day
$\mu_1$	Mean of errors in DO predictions	$-0.097 \text{ gm}^{-3}$
$\sigma_1$	Standard deviation of errors in DO predictions	$0.679 \text{ gm}^{-3}$
$\mu_2$	Mean of errors in BOD predictions	$-0.108 \text{ gm}^{-3}$
$\sigma_2$	Standard deviation of errors in BOD predictions	$0.881 \text{ gm}^{-3}$

\*The abbreviation *hr* is used to denote hours of sunlight per day.



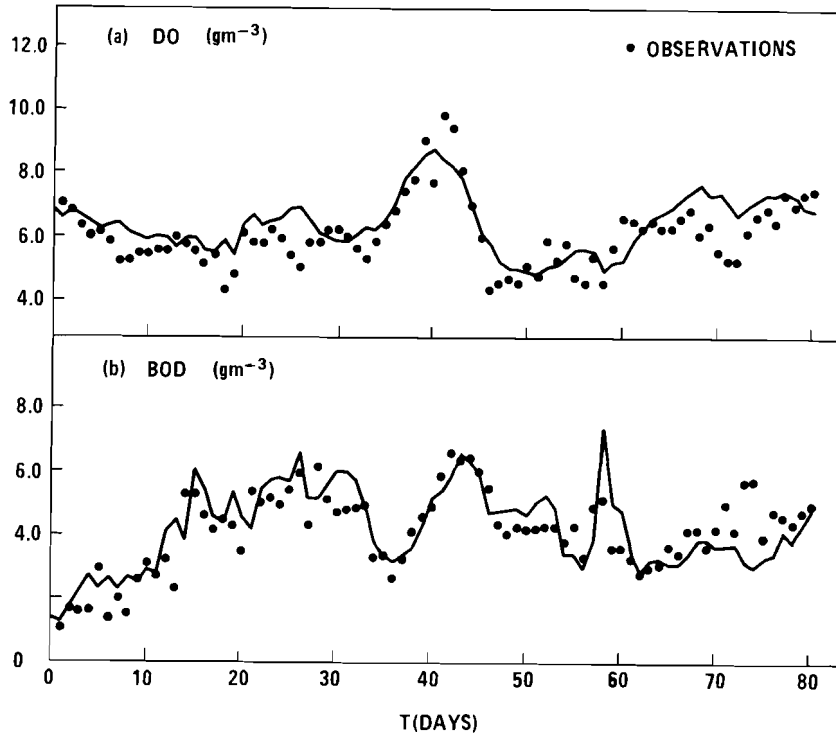


Figure 6. Deterministic model responses,  $\hat{x}(t_k)$ , and observations,  $y(t_k)$ , for Model III: (a) downstream DO concentration; (b) downstream BOD concentration.

#### 4.4. Model IV (Beck 1974, 1975)

The synthesis of Model IV depends essentially upon interpreting  $x_3(t)$  and  $x_4(t)$  in Model III, Equations (19), as

$x_3(t)$  = downstream concentration of a live algal population  
(gm<sup>-3</sup>)

$x_4(t)$  = downstream concentration of a dead algal population  
(gm<sup>-3</sup>)

and upon the assumption that algal population growth kinetics can be described by Monod (1949) kinetics with sunlight as the rate-limiting factor. For Model IV we have then

$$\begin{aligned} \text{DO: } \dot{x}_1(t) = & (\theta_1(t)/a_1)u_1(t) - (\theta_1(t)/a_1)x_1(t) + a_2(\theta_3(t) - x_1(t)) \\ & - a_3x_2(t) + a_4'(t) + a_{18}x_3(t)[u_3(t)]^{a_{19}} - a_{20}x_3(t) ; \end{aligned} \quad (21a)$$

$$\begin{aligned} \text{BOD: } \dot{x}_2(t) = & (\theta_1(t)/a_1)u_2(t) - (\theta_1(t)/a_1)x_2(t) - a_3x_2(t) \\ & + a_{21}x_4(t) ; \end{aligned} \quad (21b)$$

Live algae:

$$\dot{x}_3(t) = - (\theta_1(t)/a_1)x_3(t) + a_{22}u_3'(t)/(a_{23} + u_3'(t)) - a_{24}x_3(t) ; \quad (21c)$$

Dead algae:

$$\dot{x}_4(t) = - (\theta_1(t)/a_1)x_4(t) + a_{25}x_3(t) - a_{26}x_4(t) - a_{27} , \quad (21d)$$

where

$$u_3'(t) = u_3(t - \Delta t) , \quad (22)$$

with  $\Delta t$  being a pure time delay of one day, i.e. one sampling interval. The deterministic predictions of Model IV are shown in Figure 7. All other necessary information about the model is provided in Table 6. From both Figure 7 and Table 6 it is evident that Model IV is capable of a better representation of the observed system behavior than Model II; the most significant improvement offered by Model IV concerns the simulation of the downstream BOD response from about day  $t_{60}$  onwards--Figure 7b. Two major assumptions have been made in the derivation of Equation (21):

*Assumption (6): that no live or dead/decaying algal matter enters the reach of river across its upstream boundary.*

*Assumption (7): that the growth kinetics of the algal population in Equation (21c) are independent of the concentration of live algae.*

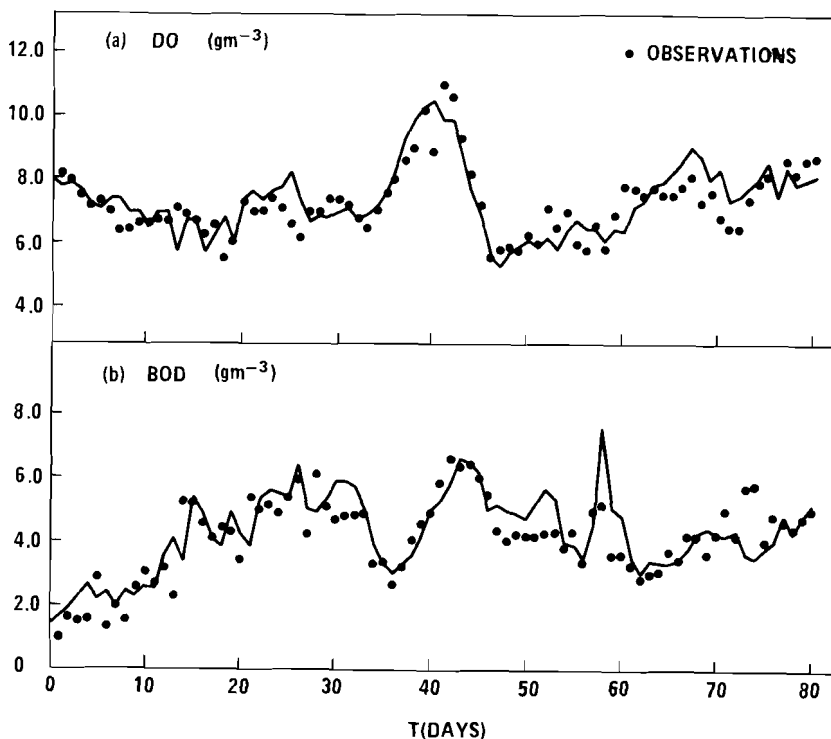


Figure 7. Deterministic model responses,  $\hat{x}(t_k)$ , and observations,  $y(t_k)$ , for Model IV: (a) downstream DO concentration; (b) downstream BOD concentration.

In practice, of course, it would be necessary to suppose that small "seed" populations of algae enter the study reach of river from upstream. The hypothesis is therefore that the nutrient-rich environment of this particular reach, provided by the sewage discharge, is capable of stimulating rapid and significant algal growth.

#### Computational Note

In Equation (21c) the following substitution is made, through Equation (22),

$$u_3(t) = u_3'(t_k - \Delta t) = u_3(t_{k-1}) \text{ for } t_k \leq t \leq t_{k+1} \quad .(23)$$

Table 6. Error statistics and additional initial conditions and parameter values for Model IV.

Variable (Parameter)	Definition	Value
$\hat{x}_3(t_0)$	Initial conditions for concentration of live algae	$0.25 \text{ gm}^{-3*}$
$\hat{x}_4(t_0)$	Initial conditions for concentration of dead algae	$0.1 \text{ gm}^{-3*}$
$a_{18}$	Rate constant for photosynthetic production of DO by live algae	$1.45^{**}$
$a_{19}$	Exponential power for dependence of algal photosynthetic DO production on sunlight conditions	$0.55^{**}$
$a_{20}$	Rate constant for respiratory consumption of DO by live algae	$2.0 \text{ (gm}^{-3} \text{ DO} \cdot \text{[gm}^{-3} \text{ algae]}^{-1} \text{ day}^{-1})$
$a_{21}$	Rate constant for BOD production by redissolved dead algal material	$4.0 \text{ (gm}^{-3} \text{ BOD} \cdot \text{[gm}^{-3} \text{ algae]}^{-1} \text{ day}^{-1})$
$a_{22}$	Maximum specific growth rate of algae	$2.1 \text{ gm}^{-3} \text{ day}^{-1}$
$a_{23}$	Saturation constant for growth rate limiting factor	$20 \text{ hrs sunlight day}^{-1}$
$a_{24}$	Specific decay rate constant for algae	$0.35 \text{ day}^{-1}$
$a_{25}$	Rate constant for production of dead algal matter from live algal matter	$1.05 \text{ day}^{-1}$
$a_{26}$	Rate constant for redissolution of dead algal material	$0.25 \text{ day}^{-1}$
$a_{27}$	Rate of sedimentation of particulate dead algal material	$0.11 \text{ gm}^{-3} \text{ day}$
$\mu_1$	Mean of errors in DO predictions	$0.158 \text{ gm}^{-3}$
$\sigma_1$	Standard deviations of errors in DO predictions	$0.643 \text{ gm}^{-3}$
$\mu_2$	Mean of errors in BOD predictions	$-0.190 \text{ gm}^{-3}$
$\sigma_2$	Standard deviations of errors in BOD predictions	$0.801 \text{ gm}^{-3}$

\*These units are merely indicative of concentration and do not in any way refer to, say,  $\text{gm}^{-3}$  dry mass of algae.

\*\*No specific units are assigned to these parameters.

It has been suggested that such a "delaying" factor may be due in part to the presence of a "stored" phase of algal population mass. The term  $\Delta t$  should not be confused with the earlier use and definition of  $\tau(t)$ , although in an abstract sense  $\Delta t$  and  $\tau(t)$  produce identical transformations of time-dependent variables. The term "stored" is perhaps more familiar in the context of microbiological aspects of sanitary engineering (see for example Busby and Andrews, 1975); it denotes a physiological state in which the nutrient has been absorbed by the cell but not yet metabolized. (Note also that the concentrations of live and dead algal populations are somewhat arbitrary; they do not, for example, have any intentional equivalence to determinations such as chlorophyll-A and dry biomass measurements.)

## 5. BLACK BOX MODELING RESULTS

### 5.1 Model Va

Recalling Assumption (4) from section 3.2, Model Va is a model that describes the behavior of the downstream DO concentration independently of the behavior of the downstream BOD concentration. In line with Equation (11) Model Va is given by

$$x_1(t_k) = \alpha_1 x_1(t_{k-1}) + \beta_{11} u_1(t_{k-1}) + \beta_{30} u_3(t_k) + \beta_{31} u_3(t_{k-1}) \quad , \quad (24)$$

which generates the response of Figure 8a--the continuous line response--with parameter values and error statistics as indicated in Table 7.

### Footnote

The parameter values for Table 7 differ slightly from those quoted elsewhere in Beck (1974; 1978a). This discrepancy results from the use of two alternative parameter estimation schemes: the Maximum Likelihood method of Åström and Bohlin (1966); and the recursive Instrumental Variable method of Young (1974). The estimates of Table 7, and similarly the estimates in Tables 8 and 9, are derived using the latter estimator. Additional details are given in Appendix 5.

### 5.2 Model Vb

It can be shown (Beck, 1976) that the parameters  $\beta_{30}$  and  $\beta_{31}$  in Equation (24) have a tendency to be nonstationary, i.e., they vary with time. Part of this time variability of the parameters can be accounted for by restating Equation (24) as

$$\begin{aligned}
 x_1(t_k) = & \alpha_1 x_1(t_{k-1}) + \beta_{11} u_1(t_{k-1}) + \beta'_{30} (\mu_{\theta_1} / \theta_1(t_k)) u_3(t_k) \\
 & + \beta'_{31} (\mu_{\theta_1} / \theta_1(t_{k-1})) u_3(t_{k-1}) \quad .
 \end{aligned}
 \tag{25}$$

In other words we are proposing that the parameters  $\beta_{30}$  and  $\beta_{31}$  in Equation (24) are not time invariant but are better represented as functions of  $\theta_1(t_k)$ , the stream discharge;  $\mu_{\theta_1}$  is a sample mean

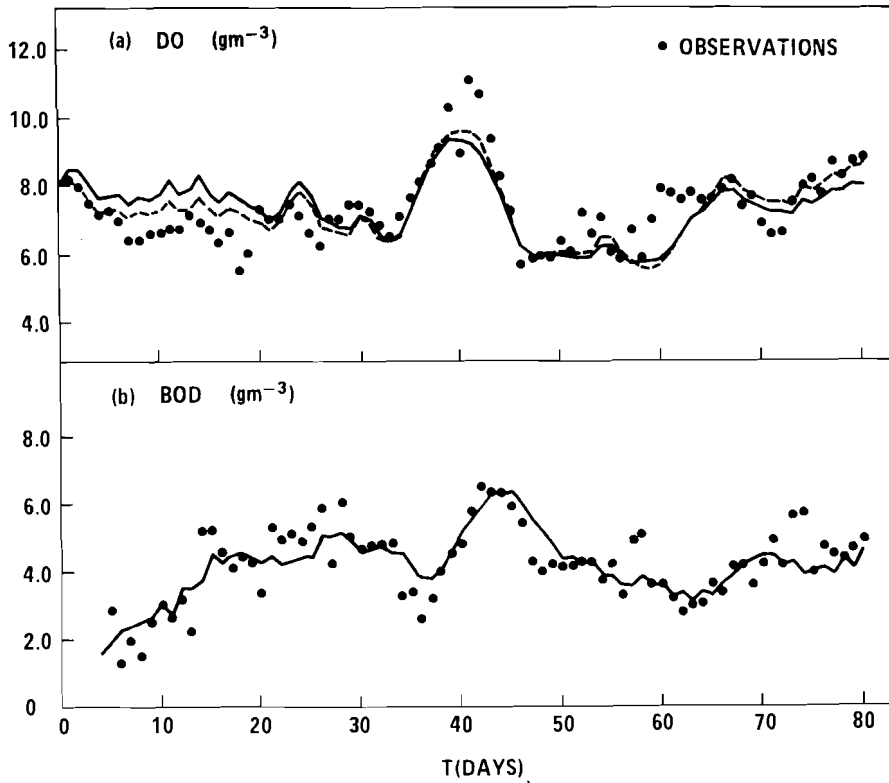


Figure 8. Deterministic model responses,  $\hat{x}(t_k)$ , and observations,  $y(t_k)$ , for: (a) downstream DO concentration, Model Va—continuous line, Model Vb—dashed line; (b) downstream BOD concentration, Model Vc.

Table 7. Parameter values, initial conditions and error statistics for Model Va.

Parameter/Variable	Value	Comments
$t_k$	$t_1$	Starting time in equation (24)
$\delta$	1	See equation (13)
$\hat{x}_1(t_0)$	$8.0 \text{ gm}^{-3}$	Initial conditions for $\hat{x}_1(t_k)$
$\alpha_1$	0.639	
$\beta_{11}$	0.229	
$\beta_{30}$	0.062	
$\beta_{31}$	0.051	
$\mu_1$	$0.016 \text{ gm}^{-3}$	Mean of prediction errors
$\sigma_1$	$0.827 \text{ gm}^{-3}$	Standard deviation of prediction errors

Table 8. Parameter values and error statistics for Model Vb.

Parameter/Variable	Value	Comments
$\mu_{01}$	$1.28 \times 10^5 \text{ m}^3 \text{ day}^{-1}$	Sample mean value for stream discharge
$\alpha_1$	0.596	
$\beta_{11}$	0.261	
$\beta'_{30}$	0.060	
$\beta'_{31}$	0.052	
$\mu_1$	$0.020 \text{ gm}^{-3}$	Mean of prediction errors
$\sigma_1$	$0.674 \text{ gm}^{-3}$	Standard deviation of prediction errors

value for  $\theta_1(t_k)$  introduced to normalize the associated expressions in Equation (25). The results of Model Vb are summarized in Figure 8a--the dashed line response--and in Table 8.

### 5.3 Model Vc

No such time variability of the parameters as identified for the DO dynamics is discernible in the corresponding black box model for the downstream BOD behavior, i.e.

$$x_2(t_k) = \alpha_1 x_2(t_{k-1}) + \beta_{21} u_2(t_{k-1}) + \beta_{32} u_3(t_{k-2}) + \beta_{34} u_3(t_{k-4}) \quad (26)$$

The results of Model Vc, which according to the error statistics show a markedly better fit to the data than any of the other models, are given in Table 9 and in Figure 8b. The value of  $\delta = 4$  in Table 9 arises because of the term  $u_3(t_{k-4})$  in Equation (26), which implies that  $x_2(t_4)$  is calculated from the measured value of  $u_3(t_0)$ .

Table 9. Parameter values, error statistics and initial conditions for Model Vc.

Parameter/Variable	Value	Comments
$t_k$	$t_4$	Starting time in equation (26)
$\delta$	4	
$\hat{x}_2(t_3)$	$1.6 \text{ gm}^{-3}$	Initial conditions for $\hat{x}_2(t_k)$
$\alpha_1$	0.826	
$\beta_{21}$	0.054	
$\beta_{32}$	0.034	
$\beta_{34}$	0.057	
$\mu_2$	$0.030 \text{ gm}^{-3}$	Mean of prediction errors
$\sigma_2$	$0.668 \text{ gm}^{-3}$	Standard deviation of prediction errors



## 6. SUMMARY OF RESULTS--SOME CRITICAL COMMENTS

The complete set of deterministic response error statistics for Models I-V are given in Table 10; in addition a relative measure is provided for the error variance as a percentage of the variance of the original time-series data.

### 6.1 Model Assessment

Since Models I through IV represent a conceptual development of DO-BOD-algae interaction models (see Beck, 1978b) it is reassuring to find that parallel with this development there runs a successive reduction of model fitting error variances. For the model representation of the DO dynamics it is apparent that Model II offers, for this particular data set, a distinct improvement over the a priori model (Model I) but that thereafter Models III and IV provide only marginal increments in accuracy. It can be concluded, nevertheless, that the more significant improvements and alterations in the description of BOD dynamics in Models III and IV do not degrade the performance of the models with respect to the DO dynamics. Model IV (the a posteriori model), in particular, reflects a concentration of effort on improving the simulated BOD responses over the final period of the data; a reward for this effort in terms of a relatively large drop in the response error variance for BOD is thus, perhaps, only to be expected.

Table 10. Survey of error statistics for Models I-V.

	DO			BOD		
	$\mu_1(\text{gm}^{-3})$	$\sigma_1(\text{gm}^{-3})$	$\sigma_1^2/\sigma_D^2 (\%)$	$\mu_2(\text{gm}^{-3})$	$\sigma_2(\text{gm}^{-3})$	$\sigma_2^2/\sigma_B^2 (\%)$
Original Data	7.282( $\mu_D$ )	1.067( $\sigma_D$ )	-	4.112( $\mu_B$ )	1.265( $\sigma_B$ )	-
Model Errors						
I	0.465	0.855	64.2	0.817	1.271	100.0
II	0.088	0.657	37.9	0.328	0.968	58.6
III	-0.097	0.679	40.5	-0.108	0.881	48.5
IV	0.158	0.643	36.3	-0.190	0.801	40.1
Va	0.016	0.827	60.1	-	-	-
Vb	0.020	0.674	39.9	-	-	-
Vc	-	-	-	0.030	0.668	27.9

A different form of model error sequence to that given in Equation (12) can be computed, which we shall call the *one-step-ahead prediction errors*. Briefly, for the internally descriptive model the computation involves imbedding the model, Equations (4) and (5), within a Kalman filter formulation from which can be generated the one-step-ahead prediction errors, or the innovations process residual errors,

$$\varepsilon'(t_k | t_{k-1}) = y(t_k) - \underline{\hat{x}}'(t_k | t_{k-1}) \quad . \quad (27)$$

Here the one-step-ahead predictions  $\underline{\hat{x}}'(t_k | t_{k-1}) = [\hat{x}'_1(t_k | t_{k-1}), \hat{x}'_2(t_k | t_{k-1})]$  of downstream DO and BOD concentrations are distinctly different from the predictions defined by Equation (10). Whereas the deterministic model predictions  $\underline{\hat{x}}(t_k)$  utilize the measured information on  $u(t_k)$  and  $\theta(t_k)$ , the one-step-ahead predictions  $\underline{\hat{x}}'(t_k | t_{k-1})$  at time  $t_k$  utilize *in addition* the measured output data up to and including  $y(t_{k-1})$ . Similarly, one-step-ahead prediction error sequences can be defined and computed for black box model representations. (Further details are given in Appendix 5.)

Table 11 offers a companion view of the modeling results based on the one-step-ahead prediction errors. Since in practice the statistical properties of these latter errors for the internally descriptive models are dependent upon certain assumptions about the statistical properties and variance-covariances of the random processes  $\xi(t)$  and  $\eta(t_k)$  in Equations (4) and (5), Table 12 adds the necessary information for associated matrix specifications in the corresponding linear Kalman filter. The overall pattern of results in Table 11 is largely a reflection of Table 10, except notably in that the BOD prediction errors for Model IV have statistical properties which are virtually identical with the statistical properties of the equivalent deterministic response errors in Table 10.

An alternative criterion for model assessment, which parallels the studies of Harleman (1978), is to reconsider the fitting error statistics in the light of the number of model parameter values that are formally estimated from the given time-series data. By "formally" here we wish to draw a clear distinction between the use of parameter estimation algorithms and the estimation of parameters by the informal method of trial and error comparisons of deterministic simulations with the field data. It is impossible, therefore, to argue against the details of Table 13 from which it is observed that hand in hand with a decrease of internally descriptive model fitting error statistics goes an increase in the number of model parameters. This is not surprising. However, the number of parameters formally "fitted" in each model is actually rather constant.

Table 11. Survey of one-step-ahead prediction error statistics for Models I-V.

	DO			BOD		
	$\mu_1' (\text{gm}^{-3})$	$\sigma_1' (\text{gm}^{-3})$	$(\sigma_1')^2 / \sigma_D^2 (\%)$	$\mu_2' (\text{gm}^{-3})$	$\sigma_2' (\text{gm}^{-3})$	$(\sigma_2')^2 / \sigma_B^2 (\%)$
Original Data	7.282 ( $\mu_D$ )	1.067 ( $\sigma_D$ )	-	4.112 ( $\mu_B$ )	1.265 ( $\sigma_B$ )	-
One-Step-Ahead Prediction Errors						
I	0.388	0.745	48.8	0.649	1.100	75.6
II	0.081	0.591	30.7	0.259	0.890	49.5
III	-0.096	0.617	33.4	-0.106	0.836	43.6
IV	0.072	0.598	31.4	-0.178	0.793	39.3
Va	0.008	0.542	25.8	-	-	-
Vb	0.001	0.524	24.1	-	-	-
Vc	-	-	-	0.003	0.628	24.6

From the foregoing analysis the performance of the black box model for BOD (Model Vc) is apparently remarkably better than that of the internally descriptive models. In the case of the black box model for DO the introduction of time-varying coefficients for Model Vb also allows a substantial improvement upon the accuracy of Model Va. Yet the very use of words like "better" and "substantial improvement" prompts the questions of how, in model assessment, one determines which model is "best" *in some sense*, upon what criteria should this judgment be based, and can we measure whether a "significant" addition of model complexity is matched by a correspondingly "significant" addition in model accuracy. Quite apart from their more fundamental implications, certain aspects of these questions may be answered by reasoning that the choice of the correct model depends upon the intended model application. On this account many would argue that black box models are particularly ill-suited to applications such as long-term planning and management or investigation of the future impact of new engineering structures on river water quality. One would certainly not advocate the use of black box models--since they tend to be data-specific--if they are to be used to make predictions outside the range of previously observed conditions, i.e., if the *sample* of fitting data does not approximate the *population* of events that are possible. For real-time forecasting (and control), however, black box models have much more attractive prospects (see for example Beck, 1978c); and as working hypotheses about observed cause/effect relationships there is much to be gained from the use of black box models (Beck, 1978b).

Table 12. Covariance matrix specifications for generation of one-step-ahead prediction errors with a linear Kalman filter (Models I + IV only); all matrices are assumed diagonal and, where applicable, time-invariant.

State	Diagonal Elements		
	$P(t_0 t_0) = E\{(\underline{x}(t_0) - \hat{\underline{x}}(t_0 t_0))(\underline{x}(t_0) - \hat{\underline{x}}(t_0 t_0))^T\}$ A Priori Estimation Error Covariance	$Q = E\{\underline{\xi}(t_k)\underline{\xi}^T(t_k)\}$ System Noise Covariance	$R = E\{\underline{\eta}(t_k)\underline{\eta}^T(t_k)\}$ Observation Noise Covariance
$x_1$ (DO)	1.0	0.4	0.2
$x_2$ (BOD)	1.0	0.4	0.4
$x_3$	0.0	0.0	-
$x_4$	0.0	0.0	-

Table 13. Numbers of formally estimated parameter values in Models I-V.

Model	No. of Parameters	No. of Parameters Formally Estimated	Estimation Algorithm and Reference	Comments
I	4	4	Extended Kalman Filter (after Jazwinski, 1970)	Algorithm used primarily for model structure identification
II	9	4	Extended Kalman Filter	Additional parameter values selected externally to the algorithm
III	11	5(+4)	Maximum likelihood (after Källström et al., 1976)	Four previously estimated parameters assumed known
IV	14	4	-	Trial and error simulation results
Va	4	4	Instrumental Variable (after Young, 1974)	
Vb	4	4	Instrumental Variable	
Vc	4	4	Instrumental Variable	

It is still useful, nevertheless, to consider other aspects of the questions posed above in a fairly general, abstract context. Most systems analysts are aware of the intuitive notion that the quality of a model is judged by some balance between model accuracy and model complexity. So to assess the models presented here on the basis of fitting error statistics alone assumes a somewhat narrow view of model assessment, especially when the sample number of observations is probably too small to lend significant meaning to such an analysis of variance. The crucial problem, of course, is the development of a more representative measure which can be applied with ease and which allows the comparison of quite differently structured models, e.g., partial differential equations, ordinary differential equations, difference equations. In this respect recent results of Maciejowski (1977) are potentially of considerable interest. By using the theory of algorithms and by borrowing ideas from algorithmic information theory Maciejowski is able to construct a measure of model "goodness" derived from a comparison of the lengths of two specially defined computer programs. The first program, or base program, simply generates a look-up table for the original data sequence. The second program embodies the algorithms that compute the set of model predictions

and it also generates a look-up table of the associated model fitting errors, i.e., the length of this second program is a function of model complexity and model accuracy. Thus the shorter the length of the candidate model's program the better is said to be the capability of that model to represent the observed process behavior. It is worth noting, then, that for the restricted case of Model Va and Vc as a joint model of DO and BOD dynamics in the Cam, Maciejowski (1977) arrives at the following conclusion: that (depending upon certain technical details of program coding) a model with equally good "predictive power" would be one that merely draws a straight line, the respective sample average values, through the downstream DO and BOD data points.

## 6.2 Accuracy of the Models as Representations of the Physical System\*

Apart from such portentous statements as the above on this particular modeling exercise, the major grounds for critical comment and appraisal concern the biological/ecological content of Model IV, this model being the end product of the analysis.

Firstly, the ecology, such as it is, is clearly naive and macroscopic in its approximation to reality. The biological processes of death, decay, and redissolution of dead algal material are, in particular, the weakest hypotheses in the model. If the dead algal material does indeed lead to the production of an additional BOD load in the river, then Model IV is better at predicting this effect over the latter period of the experiment than any of the other internally descriptive models. It is suspected that the primary factor in providing the better prediction is the inclusion of stream discharge in the mass balance for the dead algal population, although it is not evident how the effect might be related to the low flows dominant at that time. In any event, the issues of why and whether it is dead algal matter *in the river* that causes the apparently high downstream BOD's cannot be resolved on the basis of the field data for two reasons: no measurements of phytoplankton in the river are available; and algal respiration *in the five-day BOD bottle* test will equally give rise to a higher BOD measurement.

Secondly, the possibility of nitrification in the river cannot be discounted and this too might give rise to erroneously high carbonaceous BOD observations at the downstream system boundary. Evidence obtained during the experiment, however, indicates that patterns of oxygen uptake rates in the BOD test are essentially similar for samples taken from both the upstream and the downstream locations. From this it would be difficult to establish whether nitrification was or was not significant; but neither

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\*See also Appendix 3 for further comments on diurnal variations and sedimentation processes in the River Cam.

should it be concluded that nitrification is really responsible for the effects described here by the introduction of live and dead algal population balances. Later evidence from a similar experiment in 1975, an exceptionally hot summer, suggests both that the sewage works obtains a high level of nitrification and could therefore seed the river with nitrifiers, and that factors relating to the aqueous nitrogen cycle are of considerable importance in this stretch of river.

Thirdly, the form of the Monod growth rate function in Model IV, and the justification for its introduction, require careful consideration. For example, the relative magnitudes of the saturation constant,  $a_{23}$ , and the typical values for sunlight conditions imply that growth rates are in practice approximately *linearly dependent* upon  $u_3'(t)$ . Is there, therefore, any valid reason for retaining the additional complexity of the Monod function in the model? The description of algal growth kinetics is not strictly speaking that of Monod growth kinetics since it is independent of the concentration of live algae. In addition, is it feasible that the growth cycle of algae might be better approximated by the "conceptual analog" of *three*, as opposed to two, low-pass filters in series? We might hypothesize that the outputs of the three filters are equivalent to "stored", "active" and "dead" phases of the algal population where the stored algae, representing a kind of nascent cell physiological state, do not interact with the DO and BOD dynamics but have a growth rate which is a function of sunlight conditions and the concentration of the active population; otherwise the active and dead algal masses fulfil the roles of live and dead algae, respectively, as in Model IV, with the rate of production of the active state being a function of the concentration of the stored algal matter and not a function of sunlight conditions. Such hypotheses would, in principle, give some justification for the data manipulation of using  $u_3'(t)$  instead of  $u_3(t)$  in the algal growth rate function of Model IV.

Fourthly, on points of somewhat finer detail, the field data do not permit any resolution of whether the photosynthetic DO production is due to bed-attached or floating algae; the assumption here has been that it is the latter. The evidence available would certainly suggest that stream flow-rate is important in determining the amount of DO produced by photosynthesis. For instance, implicit in Model Vb--notably a black box model--is the relationship that as flow-rate decreases, the sunlight incident on the river produces a proportionately higher amount of dissolved oxygen. But beyond this kind of macroscopic cause-effect relationship it is not easy to distinguish between the relative significances of attached or floating algae, even though a corollary of the proposed relationship would be a dependence of photosynthesis rates on turbidity.

Next, in connection with more familiar aspects of DO-BOD models it can be argued that  $a_2$  and  $a_3$ , the reaeration rate and

the BOD decay rate constants respectively, should properly be accounted for as functions of flow rate (for  $a_2$  only) and temperature. The estimated evaluation of the parameter  $a_4'(t)$  in Table 3 should also be questioned. The most probable reasons for the apparently higher initial estimated rates of oxygen consumption by bottom mud deposits are as follows:

- That the BOD measurements for  $t_0 \rightarrow t_{13}$  are systematically biased, being lower than the true values of in-stream BOD concentrations (see Appendix 1);
- That the downstream DO sensor had been drifting prior to day  $t_{20}$  when it was recalibrated--there are, however, no records now available with which to check this supposition.

Finally, as mentioned in the introductory section of the paper, the approach adopted for modeling transport and mixing properties of the river reach is not the approach commonly encountered in the literature. Further to the discussion of section 3.1 it is possible that alternative approximations to the hydrodynamical regime of the river, incorporating techniques such as that outlined in Appendix 4, may give both better characterizations of the experimental data and different insights to the observed ecological/biochemical behavior. Since the fundamental philosophy underlying the development of Model IV from Model I is one that embodies a large measure of confidence in the assumptions of the a priori model, a re-examination of these assumptions would lead to a re-examination of all the subsequent models. Expectations of substantially different results, nevertheless, should perhaps not be too high. The a priori model, Model I, can be said to simulate observed behavior adequately except for certain quite specific intervals of the experiment. Thus when expressions for the sources and sink terms of Model I are cast within a different set of assumptions about transport and mixing properties of the river, the net result might only be a change in the estimated values for the associated parameters  $a_2$ ,  $a_3$ ,  $a_4(t)$ . Thereafter, our interpretations of the desired structural modifications of Model I, although not necessarily the additional parameter values, might remain essentially similar to those made here.

## 7. CONCLUSIONS

The objectives of this paper have been:

- The dissemination of field data which can be used for the verification of DO-BOD interaction models; and
- The comparison of a number of such (dynamic) models which have been derived by reference to those field data.

The opportunity has also been taken to present a summarizing and concluding statement on modeling studies with respect to the



Cam-1972 experiment. Many questions remain unresolved and it is hoped that the interest of others will be sufficiently stimulated to provide alternative answers. Some of these questions concern the following:

- The development of terms for expressing the decay, redissolution and exertion of a BOD by dead algal material;
- The relationship for growth of an algal population with sunlight conditions as a rate limiting factor;
- The possibilities for different interpretations of the observed behavior of DO-BOD interaction when different assumptions are made about the transport and dispersive properties of the reach of river.

In this context, one of the problems of working with the same set of field data over an extended period of time is that the analyst becomes blind to certain new avenues of thought.

On the other hand, any experimental data, if they are carefully collected, are worthy of a broadly based analysis which explores differing ways of identifying the basic cause/effect mechanisms governing the system's observed behavior. The problem in this context, however, is that the modeling exercise eventually approaches the limits in the accuracy and scope (i.e. the number of state variables measured) of the field data. When this limiting point has been reached, what is really required is another, better designed, and more comprehensive experiment. For the Cam such an experiment was conducted in the summer of 1975 and the associated data are currently receiving a preliminary analysis.

It is hoped that the Cam-1972 data will have some usefulness beyond the requirements of model verification studies. Perhaps this usefulness will be rather modest for the purposes of investigating operational control schemes, which, with regard to water quality management in river basins, await a number of technical developments before they can have a proper focus on reality. The area of system identification and parameter estimation is probably where the data can be used to the greatest advantage. Additional studies of the Cam-1972 data by Young and Whitehead (1977), Whitehead (1978), Ivakhnenko et al. (1977), Moore and Jones (1978) and Tong (1978) are already evidence of this. And for those less familiar with the field of system identification a set of suitable field data for analysis can provide the best means of acquiring knowledge of techniques such as Extended Kalman Filtering, Maximum Likelihood and Instrumental Variables.

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From time to time I have been asked to provide colleagues with a copy of the Cam-1972 field data. This has encouraged me in the hope that still others too will find these data useful for their own purposes. If this hope is realized then I should be pleased to receive details of any associated results.

Appendix 1

Table A1. The Cam-1972 experimental field data.

1 <sup>a</sup>	2 <sup>b</sup>	3 <sup>c</sup>	4 <sup>d</sup>	5 <sup>e</sup>	6 <sup>f</sup>	7 <sup>g</sup>	8 <sup>h</sup>	9 <sup>i</sup>	10 <sup>j</sup>
0	June 6	9.67	2.00	8.00	2.30	1.71	16.50	12.90	0.0
1	7	9.56	2.25	8.20	1.05	1.77	16.00	8.10	1.2
2	8	9.25	2.40	8.00	1.65	1.64	15.50	7.30	1.5
3	9	9.36	2.75	7.50	1.55	1.70	15.50	2.80	3.5
4	10	9.57	1.90	7.20	1.60	1.55	15.75	3.10	0.0
5	11	9.43	2.75	7.30	2.90	1.80	15.50	7.70	0.0
6	12	9.52	1.95	7.00	1.35	1.66	16.00	4.90	0.1
7	13	9.32	2.80	6.40	2.00	1.71	16.75	1.20	0.0
8	14	9.04	2.45	6.40	1.55	1.61	17.00	11.30	0.0
9	15	9.09	3.05	6.60	2.55	1.63	17.00	0.80	4.5
10	16	8.99	2.75	6.60	3.05	1.71	17.50	11.90	0.1
11	17	8.94	4.70	6.70	2.70	1.55	18.00	9.20	0.0
12	18	9.08	4.70	6.70	3.20	1.58	17.25	0.00	1.0
13	19	9.23	3.40	7.10	2.30	1.65	16.50	13.50	1.6
14	20	9.32	7.45	6.90	5.25	1.57	16.25	7.40	0.0
15	21	9.29	5.35	6.70	5.25	1.61	15.75	0.00	0.0
16	22	9.39	4.40	6.30	4.60	1.57	15.25	6.40	0.0
17	23	9.47	4.50	6.60	4.15	1.64	15.50	8.30	0.0
18	24	9.09	6.30	5.50	4.45	1.64	16.00	0.70	0.0
19	25	8.81	4.70	6.00	4.30	1.55	17.00	6.60	0.0
20	26	8.81	4.50	7.30	3.45	1.44	17.75	2.80	0.0
21	27	8.56	7.30	7.00	5.35	1.41	18.75	2.50	4.5
22	28	8.71	7.10	7.00	5.00	1.44	18.50	10.50	0.0
23	29	8.27	6.80	7.40	5.15	1.33	19.00	10.30	0.0
24	30	8.31	6.05	7.10	4.95	1.31	19.75	11.70	1.4
25	July 1	8.89	7.55	6.60	5.40	1.36	19.00	1.30	2.0
26	2	8.66	4.60	6.20	5.95	1.40	17.25	0.00	0.8
27	3	8.20	5.95	7.00	4.30	1.31	17.50	6.10	1.6
28	4	8.29	7.10	7.00	6.15	1.29	17.00	2.70	0.0
29	5	8.44	7.95	7.40	5.10	1.22	17.25	5.50	0.0
30	6	8.00	7.45	7.40	4.70	1.24	18.00	9.90	0.0
31	7	8.62	6.70	7.20	4.80	1.19	18.00	1.40	4.2
32	8	9.07	5.35	6.80	4.85	1.31	18.00	0.50	5.6
33	9	8.62	4.00	6.50	4.90	1.38	17.25	1.40	0.0
34	10	8.80	4.00	7.05 <sup>k</sup>	3.35	1.32	17.50	6.60	0.0
35	11	8.85	3.75	7.60	3.40	1.17	17.75	11.20	0.0
36	12	9.21	3.70	8.00	2.65	1.14	18.50	11.60	0.0
37	13	8.89	3.15	8.60	3.25	1.12	19.50	12.50	0.0
38	14	9.01	3.70	9.00	4.10	1.19	20.50	13.60	0.0
39	15	9.05	4.45	10.20	4.60	1.03	20.50	12.60	0.0
40	16	8.55	3.80	8.90	4.90	1.01	20.50	9.60	0.0
41	17	8.54	4.70	11.00	5.85	1.03	20.75	12.90	0.0
42	18	8.62	6.05	10.60	6.60	1.00	20.50	6.50	0.0
43	19	7.85	5.20	9.30	6.40	1.01	20.50	5.00	0.0
44	20	7.37	5.15	8.20	6.45	1.03	20.25	5.60	0.0

Table A1 (contd.)

1 <sup>a</sup>	2 <sup>b</sup>	3 <sup>c</sup>	4 <sup>d</sup>	5 <sup>e</sup>	6 <sup>f</sup>	7 <sup>g</sup>	8 <sup>h</sup>	9 <sup>i</sup>	10 <sup>j</sup>
45	21	7.67	3.40	7.20	6.00	1.03	20.25	0.00	3.9
46	22	7.48	5.45	5.60	5.50	1.03	19.75	0.00	0.9
47	23	7.47	6.10	5.80	4.35	1.15	19.75	4.40	0.5
48	24	7.61	6.55	5.90	4.05	1.12	20.25	2.00	0.0
49	25	7.38	6.25	5.80	4.25	1.05	20.50	5.60	0.0
50	26	7.37	7.35	6.30	4.20	1.03	20.00	2.20	0.0
51	27	8.00	7.55	6.00	4.20	1.01	18.75	4.50	0.0
52	28	8.17	6.40	7.10	4.30	1.01	18.00	0.10	0.5
53	29	8.12	3.65	6.50	4.30	1.03	17.75	4.40	0.0
54	30	8.00	4.95	7.00	3.80	0.94	18.75	5.70	2.7
55	31	7.49	3.55	6.00	4.30	1.07	18.50	1.60	4.1
56	Aug. 1	7.85	6.05	5.80	3.35	1.09	18.00	1.50	28.5
57	2	7.24	9.80	6.60	4.95	2.28	17.50	0.90	0.0
58	3	7.52	5.55	5.80	5.15	2.07	17.00	5.80	0.0
59	4	7.69	6.20	6.90	3.60	1.46	17.00	1.00	13.0
60	5	8.62	3.85	7.80	3.60	1.23	17.00	5.50	0.0
61	6	8.57	3.40	7.70	3.25	1.17	17.25	1.40	6.3
62	7	8.53	4.40	7.50	2.85	1.26	18.25	9.60	0.2
63	8	8.22	3.85	7.70	3.00	1.26	18.25	6.30	0.0
64	9	8.26	3.20	7.50	3.10	1.03	17.50	8.00	0.0
65	10	8.08	2.85	7.50	3.70	1.01	18.00	9.70	0.9
66	11	8.22	3.00	7.80	3.45	0.98	18.00	10.80	0.0
67	12	8.62	3.40	8.10	4.20	1.01	18.75	5.90	0.0
68	13	8.40	3.20	7.30	4.20	1.05	19.00	4.00	0.0
69	14	8.98	2.95	7.60	3.65	1.00	19.00	7.50	0.3
70	15	8.54	3.40	6.80	4.25	0.96	19.00	1.10	0.0
71	16	8.18	3.95	6.50	5.00	0.93	18.50	8.00	0.1
72	17	8.80	2.70	6.50	4.20	1.03	18.00	4.70	0.0
73	18	8.99	2.85	7.40	5.70	1.01	17.00	4.50	0.0
74	19	8.76	3.60	7.90	5.75	0.96	16.25	10.70	0.0
75	20	9.03	3.55	8.10	4.00	0.87	17.25	0.50	0.0
76	21	9.18	5.15	7.70	4.80	0.98	17.25	12.20	0.0
77	22	8.62	3.50	8.60	4.60	1.00	17.75	3.90	0.0
78	23	7.79	4.90	8.20	4.40	0.98	17.50	10.20	0.0
79	24	7.70	5.20	8.60	4.75	0.93	18.00	10.60	0.0
80	25	8.76	5.20	8.70	5.00	0.91	18.25	8.40	0.0

<sup>a</sup>Sampled data time, for modeling purposes (day).

<sup>b</sup>Date, with respect to 1972.

<sup>c</sup>Upstream DO concentration ( $\text{gm}^{-3}$ ).

<sup>d</sup>Upstream BOD concentration ( $\text{gm}^{-3}$ ).

<sup>e</sup>Downstream DO concentration ( $\text{gm}^{-3}$ ).

<sup>f</sup>Downstream BOD concentration ( $\text{gm}^{-3}$ ).

<sup>g</sup>Stream discharge ( $10^5 \text{ m}^3 \text{ day}^{-1}$ ).

Table A1 (contd.)

- <sup>h</sup>Stream temperature (°C).  
<sup>i</sup>Sunlight incident upon local area (hrs per day).  
<sup>j</sup>Rainfall in local area (mm).  
<sup>k</sup>The underlined value at  $t_{34}$  denotes a value interpolated for a missing downstream DO concentration observation.

Notes: The measured values for BOD concentration, columns 4 and 6, during  $t_0 \rightarrow t_{13}$  (inclusive) are suspected to be underestimates of the true stream BOD conditions. These measurements are derived on the basis of carrying out the five-day BOD bottle test on diluted samples of river water; for the initial period of the experiment it had been anticipated that stream BOD levels might be quite high. In the event this precautionary measure was unnecessary and subsequent comparisons of BOD's obtained from diluted and undiluted samples indicated that analyses of the diluted samples gave consistently low BOD readings. This observation is partially confirmed when BOD measurements of the sewage works effluent are substituted for other modeling purposes (see Appendix 3).

These data are available on request and consist of the following: a) three-hourly sampled measurements of upstream and downstream DO concentration, and stream temperature, starting at 12.00 hrs on June 6th and finishing at 18.00 hrs on August 25th; b) actual sampling times for upstream and downstream BOD concentrations, corrected to the nearest three-hourly sampling instant; c) once daily averaged values for the volumetric flow rate and BOD concentration of the sewage work discharge over the experimental period.

## Appendix 2

### Cross-Sectional Dimensions of the River Cam

Table A2 gives (approximately) rectangular cross-sectional dimensions for the channel of the River Cam at roughly 200 m intervals downstream from the upstream reach boundary. Two sets of dimensions at the downstream boundary are provided since at this point the channel divides to allow the main discharge to pass over a weir, while a second channel is used for navigation through a lock. The figures in parentheses denote the channel leading to the weir; the other figures represent a cross section approximating the dimensions of the two channels combined.

Note that with respect to the reach length the figure of 4.7 km has become enshrined in earlier publications; the correct figure is the one given in Table A2. Note also that for modeling

Table A2. Cross-sectional dimensions for channel of the River Cam at roughly 200 m intervals downstream from the upstream reach boundary.

Distance From Upstream Boundary (km)	Rectangular Cross Section, Breadth × Depth (m)
0.000	22.9 × 1.33
0.101	19.7 × 1.33
0.302	22.4 × 1.05
0.503	18.0 × 1.62
0.704	17.4 × 1.43
0.905	18.9 × 1.52
1.107	20.4 × 1.52
1.308	18.0 × 1.33
1.509	19.2 × 1.52
1.710	18.1 × 1.43
1.912	17.5 × 1.62
2.113	18.8 × 1.62
2.314	19.5 × 1.52
2.515	20.7 × 1.33
2.716	22.1 × 1.43
2.918	21.3 × 1.43
3.119	27.6 × 1.52
3.320	19.2 × 1.62
3.521	24.5 × 1.43
3.723	27.7 × 1.62
3.924	29.6 × 1.33
4.125	28.0 × 1.43
4.326	32.0 × 1.43
4.527	50.4 × 1.29
(4.527)	(24.2 × 1.05)

purposes, i.e. in Table 3, the value of  $1.51 \times 10^5 \text{m}^3$  has been substituted for the volume of water held in the reach; on the basis of the figures of Table A2, the volumetric holdup of water is calculated as  $1.48 \times 10^5 \text{m}^3$ . This discrepancy arises because the more accurate details of Table A2 were not computed until after a more comprehensive experiment on the Cam in 1975. All dimensions in Table A2 are, of course, subject to the assumption of a nominal head of water in the reach.

### Appendix 3

#### Notes on Previously Published Works and Some Unpublished Work

The purpose of this Appendix is to supplement the brief analysis of the models and modeling results of the main text by summarizing, in abstract form, the details of previously published works and some unpublished studies on the Cam-1972 experiment. The order of the articles listed follows the development of the subject in preference to the chronological development of the models.

1. *The Modelling of Dissolved Oxygen in a Non-Tidal Stream*  
(Beck, 1978a)

This article (written in 1975) reviews DO-BOD interaction models as they have evolved from the classical studies of Streeter and Phelps (1925). The article is somewhat restricted in terms of a literature review since it focuses attention on a sanitary engineering approach to water quality modeling; it is thus lacking in its treatment of similar lines of investigation originating from the point of view of ecology. Such ecological models as are available, however, have been generally applied to large estuarine systems rather than to smaller freshwater rivers.

2. *A Dynamic Model for DO-BOD Relationships in a Non-Tidal Stream*  
(Beck and Young, 1975)

The purpose of this paper is to present the arguments leading to the formulation of Model II (section 4.2) and to show how this model gives a better fit to the experimental field data than the a priori model, Model I (section 4.1). An interpretation of the sustained sunlight effect might be as follows. Sewage effluent entering the river just upstream of the experimental system creates a nutrient rich environment in which populations of algae may expand rapidly to significant proportions under the stimulus of longer periods of warm, sunny weather. Model verification and parameter estimation are treated in a purely deterministic framework as a matter of repeated "trial and error" simulation comparisons with the field data.

3. *Systematic Identification of DO-BOD Model Structure*  
(Beck and Young, 1976)

This paper is the statistical counterpart of paper 2 above. It concentrates on the technical problem of model structure identification as defined within the overall context of system identification and parameter estimation. Model structure identification is the process of establishing, by reference to the field data, that the model includes all the significant physical, chemical and biological relationships between variables; and further, that these relationships have the correct form, for example, the form of first order linear growth kinetics or the form of Monod growth kinetics. (The next stage of analysis, parameter estimation, would then attempt, for example, to derive accurate estimates of



either the linear growth rate constant or the saturation constant and maximum specific growth rate constant of the Monod function.) The particular method employed to solve the model structure identification problem in this paper is the Extended Kalman Filter (EFK). In fact there is no easy way of identifying a correct structure for internally descriptive dynamic models and success in the application of the EFK depends strongly on a reasonable a priori knowledge of the model parameter values, such as those estimates obtained in paper 2.

4. *The Identification of Algal Population Dynamics in a Freshwater Stream*  
(Beck, 1975)

This paper describes the results of applying Maximum Likelihood (ML) parameter estimation to Model III, an internally descriptive model, and to the black box models Va and Vc. Using arguments parallel to those for the description of microorganism cultures in the wastewater treatment processes of activated sludge and anaerobic digestion, the paper brings together both empirical evidence and the identification/estimation results for the synthesis of Model IV. Model IV itself is verified, and its parameter values are estimated, by simple trial and error deterministic simulation methods. The paper links together Model II (from papers 2 and 3) and Model IV and shows how this can be achieved through the analysis of Models III, Va and Vc.

5. *Maximum Likelihood Identification Applied to DO-BOD-Algae Models for a Freshwater Stream*  
(Beck, 1974)

This report amplifies the ML estimation results of paper 4; a summary of these results is given in Appendix 5 of the present report for comparison with the equivalent Instrumental Variable (IV)--Approximate Maximum Likelihood (AML) parameter estimates.

6. *Random Signal Analysis in an Environmental Sciences Problem*  
(Beck, 1978b)

This and the following paper are, to some extent, more concerned with the subjects of modeling, system identification, and parameter estimation, than with the subject of DO-BOD-algae interaction. They thus exploit the Cam-1972 modeling exercise as a means for making statements on these broader issues. Paper 6 emphasizes the interpretation of modeling--or more strictly speaking, model structure identification--as a procedure of repeated hypothesis testing and decisionmaking. In other words, any given model is a working hypothesis, the validity of which should ideally be tested against experimental field data. Having carried out the test, the systems analyst is then required to decide whether the hypothesis is adequate; and if the current hypothesis is inadequate, according to the criteria of the analyst, a subsequent hypothesis must be generated and also evaluated by reference to the data. In the light of this interpretation of

modeling, the paper reviews the complete conceptual process of deriving Model IV from the starting point of Model I. The paper does not enter into any detailed discussion of parameter estimation methods, nor does it attempt to formalize the notions of hypothesis testing and decisionmaking.

7. *Model Structure Identification from Experimental Data*  
(Beck 1978d)

A theme clearly emerging from the above papers is that model structure identification is a problem central to success or failure in the modeling of complex, or poorly defined, systems. This paper defines the context of model structure identification within the subject of system identification and parameter estimation--a subject which also includes the topics of experimental design, model verification, and model validation. The paper places considerable importance on the role of the EKF algorithms (see also paper 3) in model structure identification; hence the paper presents a fairly detailed statement of how to apply the algorithms and how to interpret the results thereby obtained. The paper complements the work reported in papers 3 and 6.

NOTES ON DIURNAL VARIATIONS, SEDIMENTATION, AND MODEL APPLICATIONS

A consideration of these items has been kept separate because throughout the paper attention has been directed towards field data and models which do not deal with either diurnal variations or the sedimentation of particulate material from the sewage works effluent.

Diurnal Variations

Some brief remarks on the inclusion of these effects in the models are given in paper 2. The observed features in the data can be summarized as follows:

- Distinct patterns of diurnal variations in the downstream DO concentration become established after day  $t_{30}$  and continue uninterrupted until the end of the experiment, except for the two days succeeding the thunderstorm;
- Prior to day  $t_{30}$  diurnal variations in the (downstream) DO are indistinct with an amplitude of probably little more than  $\pm 0.25 \text{ gm}^{-3}$ ;
- After day  $t_{30}$  the amplitude of the diurnal variations rises on occasion to a maximum value of  $\pm 2.00 \text{ gm}^{-3}$ ;
- At all times the diurnal variation in the upstream DO concentration, where discernible, is significantly less than the variations observed downstream.

The phase of the downstream diurnal variations, or alternatively the timing of their peak values, shows curious changes over the experimental period and perhaps therefore deserves special mention. Figure A3.1 shows a plot of the approximate intervals of the experiment during which the peak of the diurnal oscillations occurred at more or less the same time in each successive day. The only explanation offered for Figure A3.1 is that the downstream dissolved oxygen concentration reflects a complex balance between the phase of the algal photosynthetic/respiratory cycle and the phase of diurnal variations in BOD loadings imposed on the stream by the sewage works discharge. For the first half of the experiment, when the river flow-rate is steadily decreasing, the phase of the (transported) BOD loadings at the downstream boundary can be expected to change proportionately. Indeed, over this initial period the effects of algae are not dominant, whereas the mean detention time of the reach, calculated on the basis of mean volumetric holdup and the stream discharges of Appendix 1, is seen to vary from 0.9 days to 1.5 days. This is a total change of 0.6 days (14 hours) in the detention time of the reach, which, assuming a constant phase for the sewage discharge, should effect an equivalent change in the phase of the downstream DO diurnal variations, as demonstrated by Figure A3.1. For the second half of the experiment stream discharge is approximately constant, apart from the thunderstorm, and in any case the effects of algae are expected to dominate, which implies that the phase of diurnal oscillations should be roughly constant.

#### Sedimentation

In order to evaluate a model which predicts downstream variations on the basis of the sewage works effluent quality, certain strong assumptions have to be made concerning the upstream DO and BOD concentrations at the effluent outfall. These assumptions may not be so stringent in practice, however, because the effluent BOD tends to dominate upstream BOD conditions, while the upstream weir dominates the DO conditions sufficiently for the values of  $u_1$  to be substituted for the pattern of stream DO variations at the effluent outfall. Given such assumptions, the analysis reveals the following two salient features:

- That the predicted downstream BOD concentrations considerably over-estimate the observed concentrations for the period  $t_0 \rightarrow t_{13}$ ;
- That in general the model gives higher downstream BOD concentrations than expected.

We have already alluded to the first point both in section 6 of the paper and in Appendix 1; it is a consequence not so much of error in the model, but of error in the BOD measurements. If a correction for the second point is hypothesized as an increased rate constant for BOD decay,  $a_3$ , such a correction substantially degrades the performance of the model in its predictions

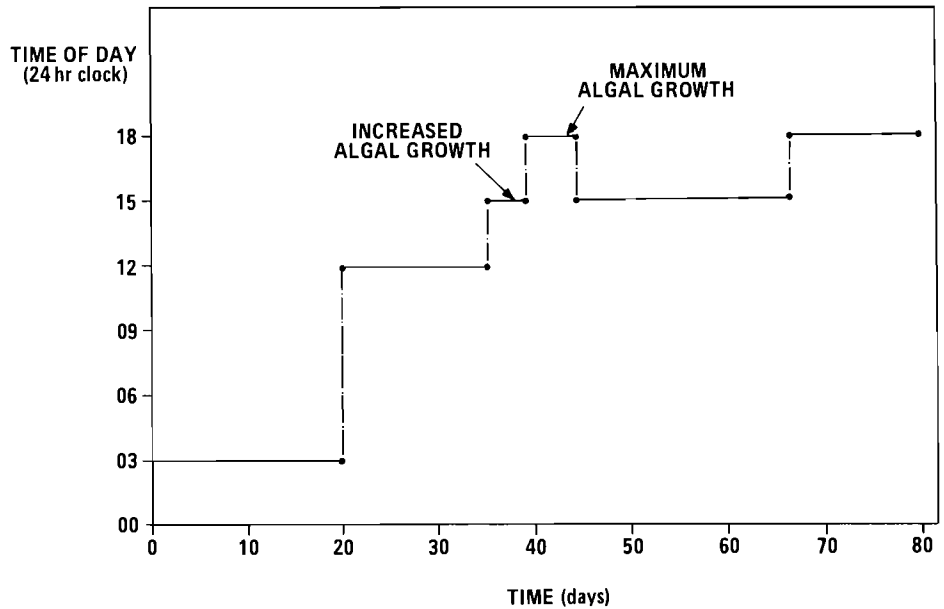


Figure A3.1. Timing peak diurnal DO concentration, on a 24-hour clock basis, for various periods of the experiment.

of the downstream DO levels. On the other hand, if the constant  $a_3$  is divided into two parts conceptually, that is  $a_{31}$  and  $a_{32}$ , say, where  $a_{31}$  (effectively the decay rate constant) fulfils the role of  $a_3$  in the DO equations of Models I through IV, and where  $a_{32}$  enters the corresponding BOD equations as a term representing sedimentation, it is possible to improve the model's BOD performance without degrading its DO performance. Suitable values for  $a_{31}$  and  $a_{32}$  are found to be  $0.32 \text{ day}^{-1}$  and  $0.16 \text{ day}^{-1}$  respectively, so that to all intents and purposes the degree of DO-BOD interaction is preserved as for the models of the paper, except that a portion of the sewage works effluent BOD, presumably that portion attached to particulate matter, settles on to the river bed. Certainly the proposal that sedimentation is significant in the short stretch of river between the effluent and the upstream weir and not significant below the weir seems plausible.

### Model Application

In section 3.1 it is stated that the originally intended application of the DO-BOD interaction models was to be in the synthesis of automatic, on-line control schemes for the day-to-day maintenance of stream DO levels. Thus, apart from the more realistic nature of a *dynamic* model as a description of a system which is rarely at a true steady state, the character of the models presented in the paper is aimed primarily at *operational*, and not design/planning, water quality management and control. (Even so, this does not necessarily preclude the use of dynamic models in the planning phases of river basin management as demonstrated by Whitehead (1976).)

There are at least three ways in which one can attempt to control the DO concentration at some point in the river system downstream of an effluent outfall. The first two of these three ways view the problem of DO control as a problem of manipulating the BOD loading placed on the receiving river by the sewage discharge in one of two ways: either one regulates the degree of BOD removal from the raw sewage; or one regulates the rate of treated sewage discharge to the stream by employing a post-treatment detention lagoon. Simulation results with Model I for case (i) and with Model II for case (ii) applied to the Cam-1972 data are reported in Young and Beck (1974). Clearly there are a number of assumptions implicit in these studies that are not valid in practice. Among the most important technical constraints on this kind of operational control are: that the degree of BOD removed from sewage cannot be varied at will from one day to the next; that the required instrumentation, telemetry/communications networks are costly, or do not exist; and that there may not be sufficient land available for the construction of a large post-treatment lagoon.

The third form of DO control, namely artificial instream aeration, is attractive for the very reason that it seems more immediately practicable. Whitehead (1977), for example, discusses such an operational control scheme using the Cam-1972 data to demonstrate his results. His model, however, while similar in some senses to Model II, is yet substantially different from all the models presented in the text; a full report of Whitehead's dynamic model for the Cam can be found in Young and Whitehead (1977).

#### Appendix 4

##### A Method for Time-Variable Transportation Delay Simulation

This Appendix describes a method of time-variable transportation delay simulation proposed by Coggan and Noton (1970); in fact it is worth noting that Coggan and Noton incorporate this form of simulation in an application of the same Extended Kalman Filtering algorithms that are used for analysis of the Cam-1972 data (Beck and Young, 1976).

The essential concepts behind the simulation are that the transportation delay element of Figure 3b can itself be imagined as a combination of  $n_a$ , say, fixed length (time-invariant) transportation delays and  $n_b$ , say, CSTR's in series (see Figure A4.1). The purpose of the time-invariant transportation delay section is to simulate the *minimum* expected transportation delay through the reach of river. (Recall that the term "transportation delay" denotes the time taken before any response is detected downstream as a consequence of any change in the upstream substance concentration.) The purpose of the multiple CSTR's is to simulate "flexibility" in the total transportation delay,  $\tau(t)$ , as it varies between the minimum,  $\tau_{\min}$ , and maximum,  $\tau_{\max}$ , expected values for the given stretch of river. Precisely how the numbers of elements  $n_a$  and  $n_b$  are chosen will be discussed below.

Suppose that we have as input to the first discrete-time delay element a concentration of (conservative) substance,  $z(t)$ , and that as output from the last CSTR element a concentration of that same substance,  $z(t - \tau(t))$ , where  $\tau(t)$  is the time-variable transportation delay referred to in the main body of the paper. (Recall also that by Assumption 2 in section 3.1 it has been assumed that materials flowing through the transportation delay behave as conservative substances.) The simulation of the total transportation delay may then be represented by

$$\begin{aligned} z_1(t_j) &= z(t_{j-1}) \\ z_2(t_j) &= z_1(t_{j-1}) \\ &\vdots \\ z_{n_a}(t_j) &= z_{n_a-1}(t_{j-1}) \end{aligned} \tag{A4.1a}$$

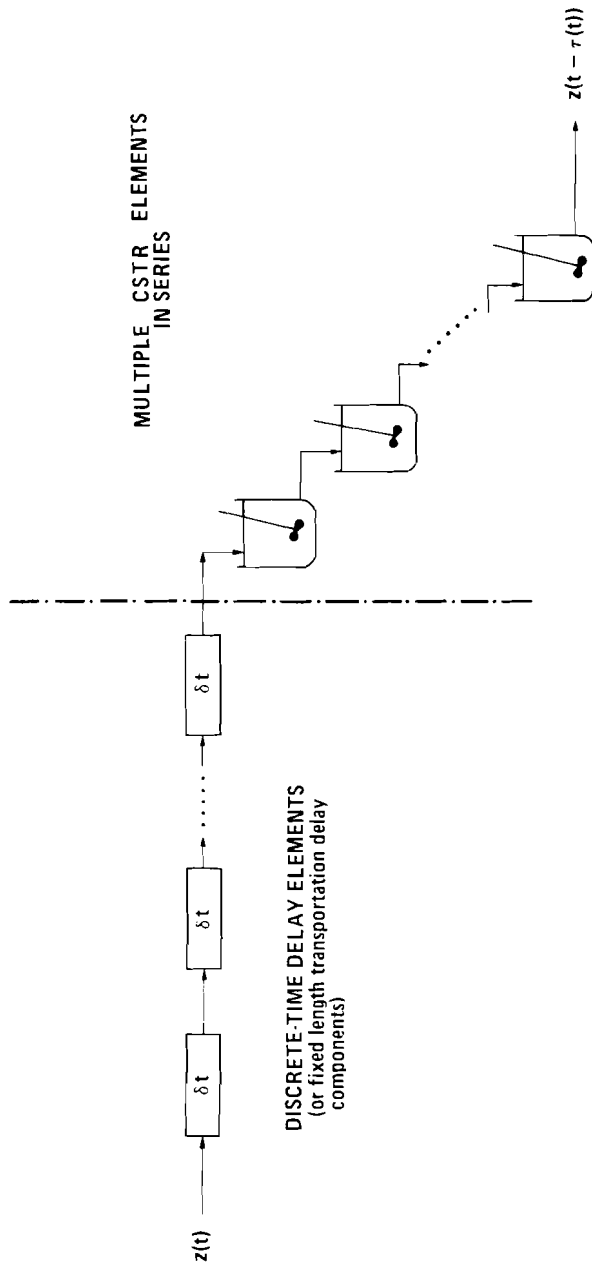


Figure A4.1. Conceptual representation of a simulation for a time-varying transportation delay.

$$\begin{aligned}
 \dot{z}_{n_a+1}(t) &= -z_{n_a+1}(t)/T(t) + z_{n_a}(t)/T(t) \\
 \dot{z}_{n_a+2}(t) &= -z_{n_a+2}(t)/T(t) + z_{n_a+1}(t)/T(t) \\
 &\vdots \\
 \dot{z}_{n_a+n_b}(t) &= -z_{n_a+n_b}(t)/T(t) + z_{n_a+n_b-1}(t)/T(t)
 \end{aligned} \tag{A4.1b}$$

with

$$z(t - \tau(t)) = z_{n_a+n_b}(t)$$

where  $z_i(t_j)$ ,  $i = 1, 2, \dots, n_a$ , is the output of the  $i$ th discrete-time delay element and  $z_i(t_j)$ ,  $i = n_a + 1, n_a + 2, \dots, n_a + n_b$ , is the output of the  $(i - n_a)$ th CSTR element. The notation of  $t_j$  to represent discrete-time instants draws a distinction between  $t_j$  and the discrete-time notation  $t_k$  of the paper. This is because the length of each time-invariant transportation delay element in the above simulation of Equations (A4.1a), that is,  $\delta t = (t_j - t_{j-1})$  may or may not be equivalent to the sampling interval  $\Delta t = (t_k - t_{k-1})$  of the measured field data. In order to match the solutions of Equations (A4.1a) at the instants  $t_j$ , Equations (A4.1b) are integrated over the intervals  $t_{j-1} \rightarrow t_j$ . Thus note how Equations (A4.1a) are connected through  $z_{n_a}$  to Equations (A4.1b); one would therefore expect the substitution of

$$z_{n_a}(t) = z_{n_a}(t_j) \text{ for } t_j \leq t \leq t_{j+1} \quad . \tag{A4.2}$$

Note further that the time constant (or mean residence time) for each CSTR element,  $T(t)$ , is time varying; with respect to Models II and III of the text, a low-pass filter is the same concept as a CSTR. The variability of  $T(t)$  is where the necessary flexibility appears in the simulation, since for a long transportation delay  $T(t)$  should be large, i.e. giving a slow response, and for a short transportation delay  $T(t)$  should be very small, i.e. giving a fast response. As with  $z_{n_a}(t)$ , for computational purposes,



$$T(t) = T(t_j) \text{ for } t_j \leq t \leq t_{j+1} \quad . \quad (\text{A4.3})$$

Looking at Equations (A4.1), there are several choices to be made in order to implement the simulation, and these concern the integration time-steps ( $t_j - t_{j-1}$ ); the number of elements  $n_a$  and  $n_b$ ; the specification of  $T(t)$  for the CSTR elements; and the computation of  $\tau(t)$ . Bearing in mind the use of the overall DO-BOD interaction model to compare model predictions with observations at time  $t_k$ , it is sensible to choose the integration time step such that the sampling interval of the data is some integer multiple,  $d$ , of this time step

$$\Delta t = d \cdot \delta t \quad . \quad (\text{A4.4})$$

Thus having defined  $\delta t$  by the choice of  $d$ ,  $n_a$  can be chosen as

$$n_a = \text{int pt } [\tau_{\min}/\delta t] \quad (\text{A4.5})$$

in which  $\text{int pt } [\cdot]$  means the integer part of the ratio between the minimum expected transportation delay,  $\tau_{\min}$ , and the integration time step. Similarly,  $n_b$  can be chosen according to (Coggan and Noton, 1970),

$$n_b = \text{int pt } [0.5(\tau_{\max}/\delta t - n_a)] + 1 - \text{int pt } [n_a \delta t / \tau_{\max}] \quad (\text{A4.6})$$

where  $\tau_{\max}$  is the expected maximum transportation delay in the reach of river. The time constant of the CSTR elements is calculated on the basis of

$$T(t) = (\tau(t) - n_a \delta t) / n_b \quad (\text{A4.7})$$

subject to the condition that

$$n_a \delta t < \tau_{\min} \quad (\text{A4.8})$$

which ensures stability of the simulation, i.e.  $T(t) > 0$  for all  $t$ . Finally, a simple, but heuristic means of computing  $\tau(t)$  is

$$\tau(t) = \tau_{\min} + \left( \frac{\theta_{1\min} - \theta_1(t)}{\theta_{1\min} - \theta_{1\max}} \right) (\tau_{\max} - \tau_{\min}) \quad (A4.9)$$

in which  $\theta_{1\min}$  and  $\theta_{1\max}$  are the stream discharges corresponding to minimum and maximum values of the transportation delay,  $\tau_{\min}$  and  $\tau_{\max}$  respectively, and where  $\tau(t)$  is to be always smaller than the mean residence time of the reach, i.e.

$$\tau(t) < (a_1/\theta_1(t)) \quad (A4.10)$$

#### An Example Simulation for the Case Study

From the data of Appendix 1 and for a value of  $1.51 \times 10^5 \text{ m}^3$  for the volumetric holdup of water in the reach, we have:

Minimum mean residence time = 0.66 day for  $\theta_{1\min} = 2.28 \times 10^5 \text{ m}^3 \text{ day}^{-1}$ ;

Maximum mean residence time = 1.74 day for  $\theta_{1\max} = 0.87 \times 10^5 \text{ m}^3 \text{ day}^{-1}$ .

If we choose  $\tau_{\min} = 0.55$  day and  $\tau_{\max} = 1.5$  day, and  $d = 2$  in Equation (A4.4), then

$$\delta t = 0.5 \text{ day}$$

and by Equation (A4.5),  $n_a = 1$ . Substituting for these figures in Equation (A4.6) gives  $n_b = 2$ .

Since both the upstream DO and BOD concentrations must be modified by a transportation delay simulation, we have for Model I of the paper the following combination of Equation (14) and Equations (A4.1),

$$\begin{aligned} \dot{x}_1(t) = & (\theta_1(t)/a_1)z_3(t) - (\theta_1(t)/a_1)x_1(t) + a_2(\theta_3(t) \\ & - x_1(t)) - a_3x_2(t) + a_4(t) \end{aligned} \quad (A4.11a)$$

$$\dot{x}_2(t) = (\theta_1(t)/a_1)z_6(t) - (\theta_1(t)/a_1)x_2(t) - a_3x_2(t) + a_5(t) \quad (A4.11b)$$

$$\left. \begin{aligned} z_1(t_j) &= u_1(t_{j-1}) \\ \dot{z}_2(t) &= -z_2(t)/T(t) + z_1(t)/T(t) \\ \dot{z}_3(t) &= -z_3(t)/T(t) + z_2(t)/T(t) \end{aligned} \right\} \quad (A4.11c)$$

$$\left. \begin{aligned} z_4(t_j) &= u_2(t_{j-1}) \\ \dot{z}_5(t) &= -z_5(t)/T(t) + z_4(t)/T(t) \\ \dot{z}_6(t) &= -z_6(t)/T(t) + z_5(t)/T(t) \end{aligned} \right\} \quad (A4.11d)$$

The differential-difference Equations (A4.11) are solved by integration over the interval  $t_j \rightarrow t_{j+1}$  with the substitutions

$$\begin{aligned} \underline{u}(t_j) &= \underline{u}(t_k) \text{ for } t_k \leq t_j \leq t_{k+1} \\ \underline{\theta}(t) &= \underline{\theta}(t_k) \text{ for } t_k \leq t \leq t_{k+1} \end{aligned} \quad (A4.12)$$

in line with the interpolation scheme used elsewhere in the paper. It might in this instance be more appropriate to make a linear interpolation for  $\underline{u}(t_j)$  and  $\underline{\theta}(t)$ . In Equations (A4.11), Equation (A4.11c) denotes the transportation delay simulation for upstream DO concentrations, and Equation (A4.11d) the same for upstream BOD concentrations.

## Appendix 5

### One-Step-Ahead Prediction Errors and Some Parameter Estimation Results

This Appendix deals first with the problem of defining one-step-ahead prediction errors for the black box model, Equations (6) and (7). Upon substituting for

$$x(t_k) = y(t_k) - v(t_k) \quad (A5.1)$$

from Equation (7) in the right-hand side of Equation (6) we obtain

$$x(t_k) = A(q^{-1})y(t_k) + \sum_{i=1}^m B_i(q^{-1})u_i(t_k) - A(q^{-1})v(t_k) \quad (A5.2)$$

Now suppose that in general the lumped, colored noise sequence  $v(t_k)$  can be modeled as the following transformation of a white noise sequence  $e(t_k)$ , say,

$$v(t_k) = \frac{D(q^{-1})}{C(q^{-1})} e(t_k) \quad (A5.3)$$

where the additional polynomials are defined as

$$\begin{aligned} C(q^{-1}) &= 1 + c_1 q^{-1} + \dots + c_n q^{-n} \\ D(q^{-1}) &= 1 + d_1 q^{-1} + \dots + d_n q^{-n} \end{aligned} \quad (A5.4)$$

and where

$$E\{e(t_k)\} = 0 \text{ and } E\{e(t_k)e(t_j)\} = 0 \text{ for } k \neq j \quad (A5.5)$$

In Equation (A5.5)  $E\{\cdot\}$  is the expectation operator. The one-step-ahead prediction  $\hat{x}'(t_k|t_{k-1})$  is then defined as

$$\hat{x}'(t_k|t_{k-1}) = A(q^{-1})y(t_k) + \sum_{i=1}^m B_i(q^{-1})u_i(t_k) - A(q^{-1})\hat{\varphi}(t_k|t_{k-1}) \quad (A5.6)$$

in which  $A(q^{-1})\hat{\varphi}(t_k|t_{k-1})$  implies estimates of past values of  $v$ , i.e.  $v(t_{k-1}|t_{k-2}), \dots$ , where

$$\hat{\varphi}(t_k|t_{k-1}) = [1 - C(q^{-1})]\hat{\varphi}(t_k|t_{k-1}) + D(q^{-1})\varepsilon'(t_k|t_{k-1}) \quad (A5.7)$$

and in which are included past values of the one-step-ahead prediction error defined by

$$\varepsilon'(t_k|t_{k-1}) = y(t_k) - \hat{x}'(t_k|t_{k-1}) \quad (A5.8)$$

A comparison of Equation (A5.6) with Equation (11) shows clearly how the deterministic model predictions and the one-step-ahead predictions differ in their utilization of the measured output information  $y(t_k)$ . The one-step-ahead prediction also includes a term that incorporates a function of the one-step-ahead prediction errors. In fact, recalling the definition of  $A(q^{-1})$  in Equation (9), Equations (A5.6) and (A5.7) define the one-step-ahead prediction  $\hat{x}'(t_k|t_{k-1})$  at time  $t_k$  to be a function of the measured output data and previous errors up to and including  $y(t_{k-1})$  and  $\varepsilon'(t_{k-1}|t_{k-2})$ . Application of the backward shift operator, Equation (8), to the one-step-ahead prediction errors gives  $q^{-1}\{\varepsilon'(t_k|t_{k-1})\} = \varepsilon'(t_{k-1}|t_{k-2})$ . Implicit in Equation (A5.7) is the assumption that the best estimate of the noise sequence,  $\hat{\varphi}(t_k|t_{k-1})$ , can be derived on the basis of the noise process model, Equation (A5.3), with  $\varepsilon'(t_k|t_{k-1})$  substituted as an approximation of  $e(t_k)$ . Thus the one-step-ahead prediction error sequences for the black box models are dependent upon the way in which the noise processes are characterized in any given model (see below).

Table A5.1 gives, for completeness, a comparison of the identified noise model structures and parameter estimates and estimation errors in the black box models when Maximum Likelihood (ML) and Instrumental Variable-Approximate Maximum Likelihood (IV-AML) estimators are used. For ML estimation the noise process model of Equation (A5.4) is necessarily constrained as

$$v(t_k) = \frac{D(q^{-1})}{[1 - A(q^{-1})]} e(t_k) \quad (A5.9)$$

that is,

$$C(q^{-1}) \equiv [1 - A(q^{-1})] \quad (A5.10)$$

and hence, for example, the parameter value  $c_1 \equiv -\alpha_1$  is inserted where appropriate in Table A5.1.

Since the ML estimation results of Table A5.1 are based on a modified data point  $u_2(t_{57}) = 6.5(\text{gm}^{-3})$ , it is probably for this reason alone that the ML and IV-AML estimates of  $\beta_{21}$  in Model Vc differ significantly. Indeed, given the relatively large estimation error for the IV-AML estimate of  $\beta_{21}$ , it is debatable whether the associated term  $u_2(t_{k-1})$  should be included in the model structure. It must be admitted, however, that the method of computing parameter estimation errors for the IV-AML estimator is only approximate. Table A5.1 indicates one further significant difference between the IV-AML and ML results, as follows. For the ML Models Va and Vc the equivalent continuous-time first-order time constants for the DO and BOD dynamics are respectively 2.98 days and 3.49 days, i.e. closely similar. For the IV-AML Models Vb and Vc the two time constants are 1.93 days and 5.23 days respectively for the DO and BOD. On the assumption that the sunlight conditions,  $u_3(t_k)$ , are providing the primary input disturbances and that it is the responses to this input, as opposed to the upstream DO or BOD conditions, that the models are preferentially estimating, then one can conclude that the IV-AML estimated models confirm findings reported elsewhere (Beck, 1975, 1978b). This observation, namely that the downstream DO concentration responds more quickly than does the downstream BOD concentration to a change in sunlight conditions, is analogous to interpretations of the role of  $u_3(t)$  in Models III and IV. However, one should perhaps not place too much emphasis on this sort of appraisal of black box model results since they may be no more meaningful than some spurious statistical property of the field data.

Table A5.1. Comparison of ML and IV-AML estimation results for Models Va, Vb, and Vc.

Model	Parameter	IV-AML	ML
Va	$\alpha_1$	$0.639 \pm 0.155$	$0.715 \pm 0.064$
	$\beta_{11}$	$0.229 \pm 0.120$	$0.174 \pm 0.050$
	$\beta_{30}$	$0.062 \pm 0.024$	$0.057 \pm 0.016$
	$\beta_{31}$	$0.051 \pm 0.027$	$0.044 \pm 0.017$
	$c_1$	$-0.644 \pm 0.036$	$(-0.715 \pm 0.064)$
	$c_2$	$-0.139 \pm 0.036$	-
	Residuals Standard Deviation	$\sigma'_1$ 0.542	0.554
Vb	$\alpha_1$	$0.596 \pm 0.123$	
	$\beta_{11}$	$0.261 \pm 0.091$	
	$\beta'_{30}$	$0.060 \pm 0.018$	
	$\beta'_{31}$	$0.052 \pm 0.022$	
	$d_1$	$-0.641 \pm 0.027$	
	Residuals Standard Deviation	$\sigma'_1$ 0.524	
Vc	$\alpha_1$	$0.826 \pm 0.078$	$0.751 \pm 0.062$
	$\beta_{21}$	$0.054 \pm 0.056$	$0.102 \pm 0.042$
	$\beta_{32}$	$0.034 \pm 0.019$	$0.048 \pm 0.015$
	$\beta_{34}$	$0.057 \pm 0.021$	$0.060 \pm 0.020$
	$c_1$	$-0.311 \pm 0.035$	$(-0.751 \pm 0.062)$
	$c_2$	$-0.160 \pm 0.035$	-
	$d_1$	-	$-0.520 \pm 0.128$
	Residuals Standard Deviation	$\sigma'_2$ 0.628	0.627

## REFERENCES

- Åström, K.J., and T. Bohlin (1966), Numerical Identification of Linear Dynamic Systems from Normal Operating Records, in P.H. Hammond, ed., *Theory of Self-Adaptive Control Systems*, Plenum, New York.
- Beck, M.B. (1974), *Maximum Likelihood Identification Applied to DO-BOD-Algae Models for a Freshwater Stream*, Report 7431(C), Division of Automatic Control, Lund Institute of Technology, Sweden.
- Beck, M.B. (1975), The Identification of Algal Population Dynamics in a Non-Tidal Stream, in G.C. Vansteenkiste, ed., *Computer Simulation of Water Resources Systems*, North-Holland, Amsterdam.
- Beck, M.B. (1976), Identification and Parameter Estimation of Biological Process Models, in G.C. Vansteenkiste, ed., *System Simulation in Water Resources*, North-Holland, Amsterdam.
- Beck, M.B. (1978a), Modelling of Dissolved Oxygen in a Non-Tidal Stream, in A. James, ed., *Mathematical Models in Water Pollution*, Wiley, London.
- Beck, M.B. (1978b), Random Signal Analysis in an Environmental Sciences Problem, *Applied Mathematical Modelling*, 2, 1, 23-29.
- Beck, M.B. (1978c), *Real-Time Control of Water Quality and Quantity*, RM-78-19, International Institute for Applied Systems Analysis, Laxenburg, Austria.
- Beck, M.B. (1978d), Model Structure Identification from Experimental Data, in E. Halfon, ed., *Theoretical Systems Ecology*, Academic, New York.
- Beck, M.B., and P.C. Young (1975), A Dynamic Model for DO-BOD Relationships in a Non-Tidal Stream, *Water Research*, 9, 769-776.
- Beck, M.B., and P.C. Young (1976), Systematic Identification of DO-BOD Model Structure, *Proc. Am. Soc. Civil Engrs., J. Env. Eng. Div.*, 102, EE5, 909-927.
- Buffham, B.A., and L.G. Gibilaro (1970), A Unified Time Delay Model for Dispersion in Flowing Media, *Chemical Engineering Journal*, 1, 31-36.
- Busby, J.B., and J.F. Andrews (1975), Dynamic Modelling and Control Strategies for the Activated Sludge Process, *J. Wat. Pollut. Control Fedn.*, 47, 1055-1080.



- Coggan, G.C., and A.R.M. Noton (1970), Discrete-Time Sequential State and Parameter Estimation in Chemical Engineering, *Trans. Instn. Chem. Engrs.*, 48, T255-T264.
- Dobbins, W.E. (1964), BOD and Oxygen Relationships in Streams, *Proc. Am. Soc. Civil Engrs., J. Sanit. Eng. Div.*, 90, 53-78.
- Harleman, D.R.F. (1978), *A Comparison of Water Quality Models of the Aerobic Nitrogen Cycle*, RM-78-34, International Institute for Applied Systems Analysis, Laxenburg, Austria.
- Himmelblau, D.M., and K.B. Bischoff (1968), *Process Analysis and Simulation*, Wiley, New York.
- Ivakhnenko, A.G., G.I. Krotov, V.I. Cheberkus, and V.N. Vysotskiy (1977), Identification of Dynamic Equations of a Complex Plant on the Basis of Experimental Data by Using Self-Organization of Models: Part 1, One-Dimensional Problems, *Soviet Automatic Control*, 10(2), 24-30.
- Jazwinski, A.H. (1970), *Stochastic Processes and Filtering Theory*, Academic, New York.
- Källström, G.C., T. Essebo, and K.J. Åström (1976), *A Computer Program for Maximum Likelihood Identification of Linear Multivariable Stochastic Systems*, preprints, IVth IFAC Symposium on Identification and System Parameter Estimation, Institute of Control Sciences, Moscow, Part 2, 508-521.
- Maciejowski, J.M. (1977), The Assessment of Rival Models of the Same Finite Set of Observations, in H. Akashi, ed., *Preprints IFAC Symposium on Environmental Systems Planning, Design and Control*, Pergamon, Oxford, 227-231.
- Monod, J. (1949), The Growth of Bacterial Cultures, *Annual Review of Microbiology*, 3, 371-394.
- Moore, R.J., and D.A. Jones (1978), Coupled Bayesian-Kalman Filter Estimation of Parameters and States of A Dynamic Water Quality Model, in Chao-Lin Chiu, ed., *Applications of Kalman Filter to Hydrology, Hydraulics, and Water Resources*, Stochastic Hydraulics Program, Dept. of Civil Engineering, Univ. of Pittsburgh.
- Rinaldi, S., R. Soncini-Sessa, H. Stehfest, and H. Tamura (1978), *Modelling and Control of River Quality*, McGraw-Hill, Düsseldorf.
- Streeter, H.W., and E.B. Phelps (1925), *A Study of the Pollution and Natural Purification of the Ohio River*, Bulletin No. 146, U.S. Public Health Service, Washington, D.C.
- Tong, R.M. (1978), *An Analysis of Fuzzy Models and a Discussion of Their Limitations*, Technical Report CUED/F - CAMS/TR 173, University Engineering Department, Cambridge, England.

- Whitehead, P.G. (1976), Dynamic Models in the Planning Role, in G.C. Vansteenkiste, ed., *System Simulation in Water Resources*, North-Holland, Amsterdam.
- Whitehead, P.G. (1977), A Dynamic Stochastic Model for Non-Tidal Rivers, in *The Bedford Ouse Study: Symposium Proceedings*, Anglian Water Authority, Huntingdon, England.
- Whitehead, P.G. (1978), *An Instrumental Variable Approach to Continuous Time Models of Dispersion and Water Quality in Non-Tidal Rivers*, Technical Report CRES AS/R10, Centre for Resource and Environmental Studies, Australian National University, Canberra.
- Whitehead, P.G., and P.C. Young (1975), A Dynamic-Stochastic Model for Water Quality in Part of the Bedford Ouse River System, in G.C. Vansteenkiste, ed., *Computer Simulation of Water Resources Systems*, North-Holland, Amsterdam.
- Young, P.C. (1974), A Recursive Approach to Time-Series Analysis, *Bulletin Institute of Mathematics and Its Applications*, 10, 209-224.
- Young, P.C., and M.B. Beck (1974), The Modelling and Control of Water Quality in a River System, *Automatica*, 10, 5, 455-468.
- Young, P.C., and P.G. Whitehead (1977), A Recursive Approach to Time-Series Analysis for Multivariable Systems, *International Journal of Control*, 25, 457-482.

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